This exam contains 8 problems with a total of 100 points. You can NOT use any notes, scratch papers, and books for the exam.

To receive more points, your arguments should be as clear and explicit as possible. In case you want to quote any standard theorems from any advanced calculus textbooks, briefly state the name (if any) or the content of the theorem which you would like to apply.

I. (10 points) Given the function $f : (0, \infty) \to (0, \infty)$ defined as

$$f(x) = \begin{cases} x, & \text{if } x \text{ is a positive irrational number.} \\ \sqrt{(1+p^2)/(1+q^2)}, & \text{if } x = \frac{p}{q}, \\ \text{where } p \text{ and } q \text{ have no common factor and } p,\ q \in \mathbf{N}. \end{cases}$$

Is f continuous at any positive irrational number? Give your reasons.

II. (10 points) Given the smooth map $f : \mathbf{R}^3 \to \mathbf{R}^3$ defined as $f(x, y, z) = (u, v, w)$, where

$$\begin{cases} u = x + y^2 + (z-1)^4 \\ v = x^2 + y + (z^3 - 3z) \\ w = x^3 + y^2 + z \end{cases}$$

Show that when (x, y, z) is close to the point $(0, 0, 1)$, we can solve x, y, z in terms of u, v, w and compute $\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} + \frac{\partial z}{\partial u}$ at the point $(u, v, w) = (0, -2, 1)$.

III. (10 points) Given a connected set $A \subseteq \mathbf{R}^n$ and a continuous function $f : A \to \mathbf{R}$. Choose any m points $x_1, x_2, \ldots, x_m$ in the set A. Show that we can find $x_0, y_0 \in A$ so that

$$f(x_0) = \frac{1}{m}\{f(x_1) + f(x_2) + \ldots + f(x_m)\}$$

and

$$|f(y_0)| = \sqrt[m]{|f(x_1)|\,|f(x_2)| \cdots |f(x_m)|}.$$

Hint: Use the Intermediate Value Theorem.

IV. (10 points) A C^2 function $f : \mathbf{R}^2 \to \mathbf{R}$ is called *harmonic* on $\mathbf{R}^2$ if we have

$$\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} = 0 \quad \text{for all} \quad (x, y) \in \mathbf{R}^2.$$

Assume that (x_0, y_0) is a strict local maximum for f on $\mathbf{R}^2$ and f is harmonic. Show that all the second derivatives of f vanish at (x_0, y_0).

V. (15 points) Let $A,\ B \subset \mathbf{R}^n$ with A compact and B closed. Assume $A \cap B = \phi$, show

(1). (10 points) There is an $\varepsilon > 0$ such that $d(x, y) = \|x - y\| > \varepsilon > 0$ for all $x \in A$ and $y \in B$.

(2). (5 points) Is above true if A and B are merely closed? If yes, prove it; if no, give an example.

VI. (15 points) Compute the double integral

$$\int\int_R (x^2 + y^2) \cdot dxdy$$

over the region R bounded by the curves: $xy = 1,\ xy = 3,\ x^2 - y^2 = 1,\ x^2 - y^2 = 4.$

VII. (15 points) Find the maximum value of $f(x, y, z) = x+y+z$ subject to the constraint conditions $x^2 + y^2 = 1,\ x + z = 1.$

VIII. (15 points) Given the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined as

$$f(x, y) = \begin{cases} \frac{x^2 y^3}{x^4 + y^4}, & \text{if } (x, y) \neq (0, 0). \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Is f differentiable at the point $(0, 0)$? Give your reasons. Hint: Compute $\frac{d}{dt}\big|_{t=0} f(t \cdot v)$ for any vector $v \in \mathbf{R}^2$.