(1) (20pts) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

(i). Find all the eigenvalues of **A**.

(ii). Find an invertible matrix **P** so that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.

(2) (15pts) Let $\mathrm{T} : \mathrm{V} \longrightarrow \mathrm{V}$ be a linear transformation. Let $Im\mathrm{T}$ denote the image of T. Suppose that $Im\mathrm{T}$ is one dimensional and $\{\mathbf{v}\}$ is a basis for $Im\mathrm{T}$.

(i). Show that if T has a nonzero eigenvalue λ, then **v** is an eigenvector of T associated to the eigenvalue λ.

(ii). Show that $\mathrm{T}^2 = 0$ if and only if T has no eigenvalue other than 0.

(3) (15pts) (i). Prove the Gauss' Lemma: Let $f(x)$ be a monic polynomial in $\mathbf{Z}[x]$. If $f(x)$ is reducible in $\mathbf{Q}[x]$ then it is reducible in $\mathbf{Z}[x]$.

(ii). Find all solutions of the integer a so that $x^3 + ax + 2$ is reducible in $\mathbf{Q}[x]$.

(4) (15pts) Let $R = \mathbf{Q}[x]$, the polynomial ring over **Q**.

(i). Show that every ideal of R is generated by a polynomial.

(ii). Show that every nonzero prime ideal is generated by an irreducible polynomial.

(iii). Use (i) to show that if $f(x), g(x) \in \mathbf{Q}[x]$ have no common factor then there are polynomials $a(x), b(x) \in \mathbf{Q}[x]$ such that $a(x)f(x) + b(x)g(x) = 1$.

(iv). Show that every nonzero prime ideal is maximal.

(5) (10pts) Describe all finite abelian groups of order 72 (Use $\mathbf{Z}_n$ to represent the cyclic group of order n.).

(6) (10pts) Prove that if a group G of order 28 has a normal subgroup of order 4, then G is abelian.

(7) (15pts)

(i). Prove that $x^6 - 2$ is irreducible in $\mathbf{Q}[x]$.

(ii). Factor $x^6 - y^6$ into a product of irreducible polynomials of $\mathbf{Q}[x, y]$.

(iii). Find a splitting field of $x^6 - 2$ over **Q**.

(iv). If K is a splitting field of $x^6 - 2$ over **Q**, then what is the degree of K over **Q**?