

Part I. True or false. You need to give a brief explanation for each problem to get credits. (10 points each).

1. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function satisfying $f(z+1) = f(z+1+i) = f(z)$ for all z . Then f is a constant function.
2. For every complex number a , there exists a sequence of complex numbers with $z_n \rightarrow 0$ as $n \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{z_n}\right) = a$.
3. Let D be a region (connected open set) in the complex plane with $0 \notin D$. Then there exists an analytic function f on D with $f'(z) = \frac{1}{z}$.
4. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an analytic non-constant function, and D be a bounded region in \mathbb{C} . Suppose that $|f(z)|$ is constant on the boundary of D . Then f has a zero in D .

Part II. In this part you need to show your work. (20 points each)

1. Given polynomial $p(z) = z^7 + \pi z^5 - z^4 + 1$.
 - (a) Show that $p(z)$ is zero free on $|z|=1$ and $|z|=3$.
 - (b) Find the number of zeros of $p(z)$ in the region $D = \{z \in \mathbb{C} ; 1 < |z| < 3\}$.
2. Calculate the following residues:
 - (a) $\text{Res}\left(\frac{1}{z^5 \sin(z^2)} ; 0\right)$
 - (b) $\text{Res}\left(\frac{e^z}{(z-a)^n} ; a\right)$, n is an integer.
3. (a) Let $R(z) = P(z)/Q(z)$ be a rational function which fixes the unit circle. Show that $z = a$ is a zero of $R(z)$ if and only if $z = \frac{1}{\bar{a}}$ is a pole of $R(z)$.
 - (b) Find the most general form of Möbius transformations which fix the unit circle.