(1) (18pts) Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

(i). Find all the eigenvalues of A.

(ii). Find an invertible matrix P so that $P^{-1}AP$ is a diagonal matrix.

(iii). Evaluate $A^4 - 7A^3 + 15A^2 - 11A$.

(2) (10pts) Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Find all possible matrix B with entries in the real number $\mathbf{R}$ such that $BA = AB$.

(3) (12pts) Let A be a 3×3 matrix over the real number $\mathbf{R}$. Prove that A is nilpotent if and only if for every n the trace of A^n is 0.

(An $n \times n$ matrix C is nilpotent if $C^k = 0$ for some integer $k \geq 1$. If $C = (c_{ij})$ then the *trace* of C is $\sum_{i=1}^{n} c_{ii}$.)

(4) (20pts)

(i) Classify up to isomorphism all groups of order 21.

(ii) Find the smallest integer n so that the permutation group S_n has an abelian subgroup of order 21. Justify your answer.

(5) (20pts)

(i) Find all possible $a, b \in \mathbf{Z}_5$ so that $x^2 + ax + b$ is irreducible in $\mathbf{Z}_5[x]$.

(ii) Find all possible $c \in \mathbf{Z}_5$ so that $x^4 + x + c$ is irreducible in $\mathbf{Z}_5[x]$.

(6) (20pts)

(i) Prove that for every natural number n the polynomial $x^n - 2 \in \mathbf{Q}[x]$ is irreducible.

(ii) Find the splitting field extension of $x^4 - 2$ over $\mathbf{Q}$.

(iii) Let F be the splitting field extension of $x^4 - 2$ over $\mathbf{Q}$. Find $[F : \mathbf{Q}]$.

(iv) Let G be the Galois group of the irreducible polynomial $x^5 - 2$ over $\mathbf{Q}$. Prove that $G \cong D_4$, the dihedral group of order 8.