(1) (18pts) Let

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} \]

(i). Find all the eigenvalues of A.

(ii). Find an invertible matrix \( P \) so that \( P^{-1}AP \) is a diagonal matrix.

(iii). Evaluate \( A^4 - 7A^3 + 15A^2 - 11A \).

(2) (10pts) Let

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \]

Find all possible matrix \( B \) with entries in the real number \( \mathbb{R} \) such that \( BA = AB \).

(3) (12pts) Let \( A \) be a \( 3 \times 3 \) matrix over the real number \( \mathbb{R} \). Prove that \( A \) is nilpotent if and only if for every \( n \) the trace of \( A^n \) is 0.

(An \( n \times n \) matrix \( C \) is nilpotent if \( C^k = 0 \) for some integer \( k \geq 1 \). If \( C = (c_{ij}) \) then the trace of \( C \) is \( \sum_{i=1}^{n} c_{ii} \).)

(4) (20pts)

(i) Classify up to isomorphism all groups of order 21.

(ii) Find the smallest integer \( n \) so that the permutation group \( S_n \) has an abelian subgroup of order 21. Justify your answer.

(5) (20pts)

(i) Find all possible \( a, b \in \mathbb{Z}_5 \) so that \( x^2 + ax + b \) is irreducible in \( \mathbb{Z}_5[x] \).

(ii) Find all possible \( c \in \mathbb{Z}_5 \) so that \( x^4 + x + c \) is irreducible in \( \mathbb{Z}_5[x] \).

(6) (20pts)

(i) Prove that for every natural number \( n \) the polynomial \( x^n - 2 \in \mathbb{Q}[x] \) is irreducible.

(ii) Find the splitting field extension of \( x^4 - 2 \) over \( \mathbb{Q} \).

(iii) Let \( F \) be the splitting field extension of \( x^4 - 2 \) over \( \mathbb{Q} \). Find \([F : \mathbb{Q}]\).

(iv) Let \( G \) be the Galois group of the irreducible polynomial \( x^6 - 2 \) over \( \mathbb{Q} \). Prove that \( G \cong D_3 \), the dihedral group of order 6.