Do all problems and show all of your work for partial credits.

1. Given the parametrized curve (helix)

$$\alpha(s) = (3\cos(\frac{s}{5}), 3\sin(\frac{s}{5}), \frac{4s}{5}), \quad s \in \mathbb{R},$$

(1) Show that the parameter s is the arc length. (10%)
(2) Determine the curvature and the torsion of α. (10%)
(3) Show that the tangent lines to α make a constant angle with the z axis. (10%)

2. Consider a regular surface S in $\mathbb{R}^3$. Let $f : S \to \mathbb{R}$ be given by $f(p) = |p - p_0|^2$, where $p \in S$ and p_0 is a fixed point of $\mathbb{R}^3$. Show that $df_p(w) = 2w \cdot (p - p_0)$, $w \in T_p(S)$. (10%)

3. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere. (10%)

4. Consider the parametrized surface (Enneper's surface)

$$x(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

and show that

(1) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0. (10\%)$$

(2) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0. (10\%)$$

(3) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}. (10\%)$$

5. Compute the Christoffel symbols for a surface of revolution parametrized by

$$x(u, v) = (v^2\cos(u), v^2\sin(u), (v + 1)^2), \quad v > 0. (20\%)$$