There are 9 questions. Question 1-7 each has 10 points. Question 8,9 each has 15 points. Please answer all questions. Below, the Euclidean space $\mathbb{R}^n$, $n \geq 1$, has the usual topology.

1. Prove that a topological space X is connected if and only if every non-empty subset $A \neq X$ of X has a non-empty boundary.

2. Let (X, d) be a metric space. The distance between two subsets A, B of X is denoted and defined by $d(A, B) = \inf\{d(a, b) |\ a \in A,\ b \in B\}$.
i) Give an example of two closed subsets A, B of $\mathbb{R}$ such that $A \cap B = \emptyset$, and $d(A, B) = 0$.
ii) Let $E, F \subset X$ be two compact sets such that $E \cap F = \emptyset$. Show that $d(E, F) > 0$.

3. True or False: The interior of a subset A of a topological space X coincides with the interior of its closure. If true, prove it. If not, give a counterexample.

4. Let X be a compact set, and Y be a Hausdorff space. Let $f : X \longrightarrow Y$ be a one-to-one surjective continuous function. Show that f is a homeomorphism.

5i) Let $A \subset \mathbb{R}^n, n \geq 1$ be an open connected set. Show that A is path-connected.
ii) Give an example of a connected set $B \subset \mathbb{R}^2$ such that B is not path-connected.

6. For $i = 1, 2, 3, \cdots$, let $U_i \subset \mathbb{R}^n$, $n \geq 1$, be a dense open set. Prove that their intersection $\bigcap_{i=1}^{\infty} U_i$ is a dense subset of $\mathbb{R}^n$.

7. Give $S^1 = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1\} \subset \mathbb{R}^2$ the subspace topology. Prove that S^1 and $S^1 \times S^1$ are not homeomorphic.

8. For $i = 1, 2$, let $a_i, b_i, c_i, d_i \in \mathbb{R}$, and $(a_i, b_i, c_i, d_i) \neq (0, 0, 0, 0)$. Let

$$E_i = \{(w, x, y, z) \subset \mathbb{R}^4 |\ a_i w + b_i x + c_i y + d_i z = 0\}.$$

Suppose that $E_1 \neq E_2$. Show that the following three topological subspaces of $\mathbb{R}^4$: $\mathbb{R}^4 \setminus E_1$, $\mathbb{R}^4 \setminus (E_1 \cup E_2)$, and $\mathbb{R}^4 \setminus (E_1 \cap E_2)$ are not homeomorphic to each other.

9. Let (X, d) be a compact metric space. Let $f : X \longrightarrow X$ be a continuous map. We say that f is an isometry if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that if f is an isometry, then f is surjective.