國立中正大學九十二學年度碩士班招生考試試題

系所別：數學系　　　　　　　　　　科　目：高等微積分

This exam contains 7 problems with a total of 100 points. Provide detailed proofs to all problems.

1. (10 points) Given the function $f:(0,\infty)\to(0,\infty)$ defined as

$$f(x)=\begin{cases}0 & \text{if } x \text{ is an irrational number}\\ \frac{1}{q} & \text{if } x=\frac{p}{q}, \text{ where } p \text{ and } q \text{ have no common factor}, \quad p,\ q\in\mathbb{N}.\end{cases}$$

Is f continuous at any irrational number? Give your reasons.

2. (15 points) Given the smooth map $f:\mathbb{R}^3\to\mathbb{R}^3$ defined as $f(x,y,z)=(u,v,w)$, where

$$\begin{cases}u=x+y^2+(z-1)^4\\ v=x^2+y+(z^3-3z)\\ w=x^3+y^2+z\end{cases}$$

Show that when (x,y,z) is close to the point $(0,0,1)$, we can solve x,y,z in terms of u,v,w and compute $\frac{\partial x}{\partial v}+\frac{\partial y}{\partial v}+\frac{\partial z}{\partial v}$ at the point $(u,v,w)=(0,-2,1)$.

3. (15 points) Let $A,\ B\subset\mathbb{R}^n$ with A compact and B closed. Assume $A\cap B=\phi$, show

   (a) (10 points) There is an $\varepsilon>0$ such that $d(x,y)\geq\varepsilon>0$ for all $x\in A$ and $y\in B$.

   (b) (5 points) Is above true if A and B are merely closed? If yes, prove it; if no, give an example.

4. (15 points) Given the function $f:\mathbb{R}^2\to\mathbb{R}$ defined as

$$f(x,y)=\begin{cases}\frac{x^2y^3}{x^4+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Is f differentiable at the point $(0,0)$? Give your reasons.

5. (15 points) It is known that the improper double integral $\iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy$ has value equal to π. Use it and a suitable change of variables to evaluate the double integral

$$\iint\limits_{\mathbb{R}^2} e^{-(3x^2+3y^2-2xy)}dxdy.$$

6. (15 points)

   (a) (5 points) State the definition of uniform convergence.

   (b) (10 points) Let $f_n(x)=x^n$ and $f(x)=0$, where $x\in[0,1)$. Show that f_n converges uniformly to f on the interval $[0,a]$ for any $0<a<1$ and that f_n does not converge uniformly to f on the interval $[0,1)$.

7. (15 points) Use Lagrange Multipliers Method to find the minimum values of the function $f(x,y)=x^2-4xy+4y^2$ on the circle $x^2+y^2=1$.