Algebra and Linear Algebra

Instruction. There are seven problems.
Notations. The symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{F}_q$ stand for, respectively, the ring of integers, the field of rational, real, and complex numbers, and a finite field with q elements. The letter x denotes an indeterminate.

1. (10%) Classify groups of order 55 up to isomorphism.

2. (15%) Let $G = \mathrm{GL}_3(\mathbb{F}_2)$ be the multiplicative group of 3×3 invertible matrices with entries in $\mathbb{F}_2$. It is known that this is a simple group of order 168.

   (a) How many elements of order 7 are there in G?

   (b) Find an element $a \in G$ of order 7. Let A be the subgroup generated by a. Let $N_G(A) := \{g \in G \mid gAg^{-1} = A\}$ be the normalizer and $C_G(A) := \{g \in G \mid gbg^{-1} = b \text{ for all } b \in A\}$ be the centralizer. Find the order of $N_G(A)$ and of $C_G(A)$.

3. (20%) Let R be a ring. An element $a \in R$ is called *nilpotent* if $a^n = 0$ for some positive integer n. The subset consisting of all nilpotent elements is called the *nilradical* of R.

   (a) Prove that the nilradical is an ideal if R is commutative.

   (b) Find the nilradical of the matrix ring $M_2(\mathbb{F}_2)$.

4. (15%)

   (a) Prove that $\mathbb{Q}[x]/(x^4+1)$ is a field. [*Hint.* Eisenstein's criterion.]

   (b) Is $\mathbb{F}_{2003}[x]/(x^4+1)$ a field? Why?

5. (20%)

   (a) Find the splitting field $E \subset \mathbb{C}$ of the polynomial x^5+1 over $\mathbb{Q}$. What is the degree $[E:\mathbb{Q}]$?

   (b) Determine the Galois group of E over $\mathbb{Q}$. You must describe explicitly the action of the Galois group on E and also describe it as an abstract group.

6. (10%) Let $A = \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \\ -4 & 0 & -2 \end{pmatrix}$. Diagonalize A, i.e., find matrices P and D so that $D = P^{-1}AP$, where P is invertible and D is diagonal.

7. (10%) Let T be a linear transformation on a 4-dimensional vector space V over $\mathbb{R}$. Suppose T has minimal polynomial $x^3(x-1)$ and the eigenspace associated with the eigenvalue 0 is spanned by $(1,1,0,2)$, $(2,4,0,5)$, and $(2,0,0,3)$. Find the rank and nullity of T. Is T diagonalizable?