Algebra and Linear Algebra

(1) **(12 Points)** Is there a matrix A in $M_{3\times 5}(\mathbb{Z}_2)$ so that its reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and the first, second and fourth columns of A are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

respectively. Explain in Detail.

(2) **(12 Points)** Let $A \in M_5(\mathbb{R})$ be an idempotent matrix, that is, $A^2 = A$. Find the number of all possible non-similar Jordan forms for A.

(3) **(12 Points)** Let H and K be normal subgroups of a group $(G, \cdot, e)$ such that $H \cap K = \{e\}$. Show that $HK \simeq H \times K$.

(4) **(12 Points)** Let $\varphi\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ be the $\mathbb{Z}$-linear map sending (a, b, c) to $(b + 3c, -a + 2b, a - c)$. Find the matrix of this map with respect to the standard basis

$$\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}.$$

Is φ an isomorphism? Explain your answer.

(5) **(12 Points)** Are the roots of $x^3 - 10x^2 + 12x - 3 = 0$ in $\mathbb{C}$ constructible using straightedge and compass?

(6) **(20 Points)** Let $R = K[x, y]$ be the polynomial ring of two variables over a field K. Show that $x^3 + y^3 + 1$ is irreducible in R when $K = \mathbb{C}$. Can you find a field K so that $x^3 + y^3 + 1$ is reducible in R?

(7) **(20 Points)** Let the characteristic of a commutative ring R be a prime integer p.

(a) Show that one can think of R as a $\mathbb{Z}_p$-vector space in a natural way.

(b) Suppose $\dim_{\mathbb{Z}_p} R = n < \infty$. Find $|R|$.

(c) For each positive integer n, construct an integral domain R such that $\dim_{\mathbb{Z}_p} R = n$. You need to explain why your construction works. (You may assume that the algebraic closure of $\mathbb{Z}_p$ exists.)