Algebra and Linear Algebra

(1) (12 Points) Is there a matrix $A$ in $M_{3 \times 5}(Z_2)$ so that its reduced echelon form is
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]
and the first, second and fourth columns of $A$ are
\[
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]
respectively. Explain in detail.

(2) (12 Points) Let $A \in M_5(\mathbb{R})$ be an idempotent matrix, that is, $A^2 = A$. Find the number of all possible non-similar Jordan forms for $A$.

(3) (12 Points) Let $H$ and $K$ be normal subgroups of a group $(G, \cdot, e)$ such that $H \cap K = \{e\}$. Show that $HK \cong H \times K$.

(4) (12 Points) Let $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the $\mathbb{Z}$-linear map sending $(a,b,c)$ to $(b + 3c, -a + 2b, a - c)$. Find the matrix of this map with respect to the standard basis
\[
\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}.
\]
Is $\varphi$ an isomorphism? Explain your answer.

(5) (12 Points) Are the roots of $x^3 - 10x^2 + 12x - 3 = 0$ in $\mathbb{C}$ constructible using straightedge and compass?

(6) (20 Points) Let $R = K[x, y]$ be the polynomial ring of two variables over a field $K$. Show that $x^3 + y^3 + 1$ is irreducible in $R$ when $K = \mathbb{C}$. Can you find a field $K$ so that $x^3 + y^3 + 1$ is reducible in $R$?

(7) (20 Points) Let the characteristic of a commutative ring $R$ be a prime integer $p$.

(a) Show that one can think of $R$ as a $\mathbb{Z}_p$-vector space in a natural way.

(b) Suppose $\dim_{\mathbb{Z}_p} R = n < \infty$. Find $|R|$.

(c) For each positive integer $n$, construct an integral domain $R$ such that $\dim_{\mathbb{Z}_p} R = n$. You need to explain why your construction works. (You may assume that the algebraic closure of $\mathbb{Z}_p$ exists.)