This exam contains 5 problems with a total of 100 points.

To receive more partial credits, your arguments should be as clear and explicit as possible. You can quote any standard theorem from any advanced calculus textbooks. If there is a name for the theorem quoted, state its name

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1. (20 points) It is known that the improper double integral $\iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)} \cdot dxdy$ has value equal to π. Use it and a suitable change of variables to evaluate the following double integral

$$\iint\limits_{\mathbb{R}^2} e^{-(3x^2+3y^2-2xy)}dxdy.$$

Hint: Find a change of variables so that when you substitute the new variables (ξ, η) in $3x^2+3y^2-2xy$, you do NOT have the cross term $\xi\eta$.

2. (20 points) Let $M(2)$ denote the space of all 2×2 real matrices (it can be identified with $\mathbb{R}^4$) and let $f : M(2) \to \mathbb{R}$ be a map defined as $f(A) = \det A$, the determinant of the 2×2 real matrix A. Compute $df_I(H)$, which is the derivative of f at the point $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ evaluated at the matrix $H = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$, $h_1, h_2, h_3, h_4 \in \mathbb{R}$. For your answer, also show that $df_I(H) : H \in M(2) \to \mathbb{R}$ is a linear transformation in H.

3. (20 points) Define the funtion f in $\mathbb{R}^2$ by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

Find the critical points of f and determine whether they are local maxima, local minima or saddle points.

4. (20 points) For $n = 1, 2, 3, 4, \ldots\ldots, x \in (-\infty, \infty)$, we let

$$f_n(x) = \frac{x}{1+nx^2}.$$

Show that $\{f_n(x)\}_{n=1}^{n=\infty}$ converges uniformly on $(-\infty, \infty)$ to a function $f(x)$. Find the function $f(x)$ and also show that the relation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

5. (20 points) Let $f(x) : \mathbb{R} \to \mathbb{R}$ and $G(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be two C^2 functions such that $G(x, f(x)) = 0$ for all $x \in \mathbb{R}$. Show that we have

$$\frac{d^2y}{dx^2} = \frac{-1}{\left(\frac{\partial G}{\partial y}\right)} \left[\frac{\partial^2 G}{\partial x^2} + 2\frac{dy}{dx}\frac{\partial^2 G}{\partial x \partial y} + \left(\frac{dy}{dx}\right)^2 \frac{\partial^2 G}{\partial y^2}\right],$$

where $y = f(x)$ and the partial derivatives of G are evaluated at $(x, f(x))$.