

1. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$. (20%)

(a) Show that $\text{rank}(AB) \leq \text{rank}(A)$.

(b) Is it true $\text{rank}(AB) \leq \text{rank}(B)$? Explain.

(c) Show that $\text{rank}(AB) = \text{rank}(B)$ if and only if $\text{range}(B) \cap \text{null}(A) = \{0\}$, where $\text{range}(B)$ is the column space of B and $\text{null}(A)$ is the nullspace of A . (Hint: For any $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) + \text{nullity}(A) = n$).

2. Let $\mathbf{v}_1 = [1, -1, 1, 1]$, $\mathbf{v}_2 = [3, -1, 2, 2]$ and $\mathbf{v}_3 = [4, -2, 0, 2]$ be column vectors of matrix $A \in \mathbb{R}^{4 \times 3}$. Find an orthogonal matrix $Q \in \mathbb{R}^{4 \times 3}$ ($Q^T Q = I$) and an upper triangular matrix R such that $A = QR$. Find the projection of $\mathbf{b} = [5, -4, 3, -4]$ on the subspace $W = \text{sp}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find the minimal value of the least squares problem $\min_{\mathbf{x} \in \mathbb{R}^3} \|A\mathbf{x} - \mathbf{b}\|$. (20%)

3. Let P_2 be the space of all polynomials of degree less than or equal to 2 and

$T: P_2 \rightarrow P_2$ be the linear transformation defined by

$$T(p(x)) = p(x-2) + 2p'(x). \quad (20\%)$$

(a) Find the matrix representation A of T relative to the ordered basis $B = (x^2, x, 1)$

(b) Find an invertible matrix C such that the matrix $J = C^{-1}AC$ is a Jordan canonical form for the matrix A in (a).

(c) Compute $\underbrace{T \circ T \circ T \cdots \circ T}_{50 \text{ terms}}(3 - 2x - x^2)$.

4. Let \mathbf{u} and \mathbf{v} be nonzero vectors in \mathbb{R}^n and $\mathbf{u}^T \mathbf{v} \neq 0$. Show that the characteristic polynomial of the matrix $A = \mathbf{u}\mathbf{v}^T$ is $\det(A - \lambda I) = (-1)^n \lambda^{n-1} (\lambda - \mathbf{u}^T \mathbf{v})$. (10%)

5. Let A be an $n \times n$ matrix. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively. Prove that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent. (10%)

6. Solve the linear differential system (10%)

$$\begin{cases} \frac{dy_1}{dt} = 2y_1 + 2y_2 \\ \frac{dy_2}{dt} = 3y_1 + y_2 \end{cases}, \text{ with initial condition } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

7. Evaluate $\text{trace}(A^5 + AB - BA + 2B^2)$, where (10%)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3 & 1 & 1 & -4 \\ 2 & -1 & -1 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$