Entrance Exam of Advanced Calculus (2 pages)
Total: 100 pts

1. Let \( f(x) = x^2, \forall x \in \mathbb{R} \). Use the \( \epsilon - \delta \) argument to prove that \( f \) is a continuous function on \( \mathbb{R} \). Hint: \( \lim_{x \to 0} f(x) = 1 \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 \) such that \( |f(x) - 1| < \epsilon \) for \( |x| < \delta \). (5 pt)

2. For \( f : [0, 1] \to \mathbb{R} \) is a bounded and increasing function, we may define

\[
D_f = \{ x \in [0, 1] : \lim_{t \to x^+} f(t) \neq f(x) \}.
\]

(i) Is there a bounded and increasing function \( f \) such that the associated set \( D_f \) has infinitely many points? If yes, give an example. Otherwise, prove your answer. (5 pt)

(ii) Let \( f : [0, 1] \to \mathbb{R} \) be a bounded and increasing function. Suppose that \( \limsup_{t \to x^+} f(t) - \liminf_{t \to x^+} f(t) \geq \frac{1}{2} \), \( \forall x \in D_f \). Can \( f \) be Riemann integrable? Prove or disprove your answer. (10 pt)

3. Let \( E \) be a bounded subset of \( \mathbb{R}^n \) and \( f : E \to \mathbb{R} \) be a bounded and continuous function.

(i) Can \( f \) have a minimal point in \( E \)? If yes, prove it. If no, add an extra condition on \( E \) such that \( f \) has a minimal point in \( E \). Prove your statement. (10 pt)

(ii) Can \( f \) be uniformly continuous? If yes, prove it. If no, add an extra condition on \( E \) such that \( f \) is uniformly continuous on \( E \). Prove your statement. (10 pt)

4. Let \( \{f_n\} \) be a sequence of differentiable real-valued functions on \( (0, 1) \) such that \( \{f_n\} \) converges to \( f \) uniformly on \( (0, 1) \), where \( f \) is differentiable on \( (0, 1) \). Can \( \{f_n\} \) converge to \( f' \) pointwise on \( (0, 1) \)? Here \( f' \) means the differentiation. Prove or disprove your answer. (10 pt)
5. Let $f : A \rightarrow \mathbb{R}$ be a continuous function such that the partial derivatives $f_x$ and $f_y$ exists on $A$, where $A = (0, 1) \times (0, 1)$.

(i) Can $f$ be differentiable on $A$? Prove or disprove your answer. (5 pt)

(ii) Suppose that $f_x$ and $f_y$ are continuous on $A$. Can $f$ be differentiable on $A$? Prove or disprove your answer. (10 pt)

6. Let $f(x, y) = \sin x \cos y$, $\forall x, y \in E$, where $E = (-\pi, 0) \cup (0, \pi)$. Can $f$ have critical points in its domain $E \times E$? If yes, find them and determine whether they are local minimum, local maximum or saddle points. If no, modify the set $E$ such that $f$ has local minimum, local maximum and saddle points in its domain $E \times E$. (10 pt)

7. Calculate the following integrations.

(i) $\int_0^\infty e^{-x^2} \, dx = ?$ (10 pt)

(ii) $\int_0^\infty x^2 e^{-x^2} \, dx = ?$ (5 pt)

8. Let $f : B \rightarrow \mathbb{R}$ be a smooth function, where $B = \{(x, y) : x^2 + y^2 < 1\}$. Suppose that $f_{xx} + f_{yy} = 0$ in $B$, where $f_{xx}$ and $f_{yy}$ are the 2nd order partial derivatives of $f$ in $x$ and $y$ directions, respectively. Can $\int_B f(x, y) \, dx \, dy = \pi f(0)$? Prove or disprove your answer. (10 pt)