國立中正大學九十一學年度碩士班招生考試試題

系所別：應用數學研究所　　　　科　目：高等微積分

共一頁

Advanced Calculus

INSTRUCTION. (1) There are 11 questions (and one page.) in this problem set. (2) The total is 100 points. (3) *Label each problem clearly. Cross out the abandoned arguments.* (4) To earn partial credits, you must provide reasonable and potential arguments. You may quote any results in the suggested references.

1. (9 pts) Give a subset of $\mathbb{R}$ that has infinitely many accumulation points but contains none of them. *Verify your example!*

2. (9 pts) Let $A,\ B \subset \mathbb{R}$ be closed sets. Does
$$A + B = \{x + y \mid x \in A \text{ and } y \in B\}$$
have to be closed? *Prove or disprove!*

3. (9 pts) Let $A \subset \mathbb{R}^n$ be uncountable. Prove that A has an accumulation point.

4. (9 pts) Is a closed and bounded subset of a metric space compact? *Justify your answer!*

5. (9 pts) Show that
$$f(x) = \sqrt{x} : [0, \infty) \to [0, \infty)$$
is uniformly continuous.

6. (9 pts) Is the product of two uniformly continuous functions again uniformly continuous? *Justify your answer!*

7. (9 pts) Let $f_k(x) : [0, 1] \to [0, \infty)$ be a sequence of continuous functions such that (a) $f_k \to 0$ pointwise and (b) $f_k(x) \leq f_l(x)$ for $k \geq l$.

   Prove that $f_k \to 0$ uniformly.

8. (9 pts) Show that
$$\sum_{1}^{\infty} \frac{\sin nx}{n}$$
converges uniformly on $[\delta, 2\pi - \delta]$, where $0 < \delta < \pi$.

9. (9 pts) A function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree m, i.e., $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

   If f is differentiable, show that
$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = m f(x)$$
for $x \in \mathbb{R}^n$.

10. (9 pts) Show that
$$\int_{1}^{\infty} \frac{\sin x}{x} dx$$
is conditionally but not absolutely convergent.

11. (10 pts) Prove that any closed set $A \subset M$ is an intersection of a countable family of open sets. ((M, d) is a metric space.)