Advanced Calculus

**INSTRUCTION.** (1) There are 11 questions (and the page) in this problem set. (2) The total is 100 points. (3) Label each problem clearly. Cross out the abandoned arguments. (4) To earn partial credits, you must provide reasonable and potential arguments. You may quote any results in the suggested references.

1. (9 pts) Give a subset of \( \mathbb{R} \) that has infinitely many accumulation points but contains none of them. Verify your example!

2. (9 pts) Let \( A, B \subseteq \mathbb{R} \) be closed sets. Does
   \[ A + B = \{ x + y \mid x \in A \text{ and } y \in B \} \]
   have to be closed? Prove or disprove!

3. (9 pts) Let \( A \subseteq \mathbb{R}^n \) be uncountable. Prove that \( A \) has an accumulation point.

4. (9 pts) Is a closed and bounded subset of a metric space compact? Justify your answer!

5. (9 pts) Show that
   \[ f(x) = \sqrt{x} : [0, \infty) \to [0, \infty) \]
   is uniformly continuous.

6. (9 pts) Is the product of two uniformly continuous functions again uniformly continuous? Justify your answer!

7. (9 pts) Let \( f_k(x) : [0, 1] \to [0, \infty) \) be a sequence of continuous functions such that (a) \( f_k \to 0 \) pointwise and (b) \( f_k(x) \leq f_{k+1}(x) \) for \( k \geq 1 \).

   Prove that \( f_k \to 0 \) uniformly.

8. (9 pts) Show that
   \[ \sum_{n=1}^{\infty} \frac{\sin nx}{n} \]
   converges uniformly on \([\delta, 2\pi - \delta]\), where \( 0 < \delta < \pi \).

9. (9 pts) A function \( f : \mathbb{R}^n \to \mathbb{R} \) is homogeneous of degree \( m \), i.e.,
   \[ f(tx) = t^m f(x) \]
   for all \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \).

   If \( f \) is differentiable, show that
   \[ \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mf(x) \]
   for \( x \in \mathbb{R}^n \).

10. (9 pts) Show that
    \[ \int_{1}^{\infty} \frac{\sin x}{x} \, dx \]
    is conditionally but not absolutely convergent.

11. (10 pts) Prove that any closed set \( A \subseteq M \) is an intersection of a countable family of open sets. \( (M, d) \) is a metric space.)

    \[ P \]