共九題　總分100分

1. (10 pts) Let $\{x_n\}$ be a sequence that satisfies the condition

$$|x_n - x_{n+1}| \leq \frac{1}{2^n} \qquad n = 1, 2, 3, \ldots$$

Prove that $\{x_n\}$ is a Cauchy sequence.

2. (10 pts) Suppose that f is twice differentiable on (0,1) and that there are points $x_1 < x_2 < x_3$ in (0,1) so that $f(x_1) > f(x_2)$, $f(x_3) > f(x_2)$. Prove that there is a point $c \in (0,1)$ such that $f''(c) > 0$.

3. (10 pts) Suppose that $f \geq 0$ and is continuous on [0,1]. If

$$\int_0^1 f(x)\, dx = 0,$$

prove that $f(x) = 0$ for all $x \in [0,1]$.

4. (10 pts) (a) Prove that there exist functions $u(x,y)$, $v(x,y)$, $w(x,y)$, and $r > 0$ such that u, v, w are continuously differentiable in $B_r(1,1)$, and $u(1,1) = 1$, $v(1,1) = 1$, $w(1,1) = -1$, and satisfy the equations

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1. \end{aligned}$$

(b) Find $u_x(1,1)$, $v_x(1,1)$ and $w_x(1,1)$.

5. (10 pts) Find the maximum and minimum of $f(x,y) = x^2 + 2x - y^2$ on the set $\{(x,y) : x^2 + y^2 \leq 4\}$.

6. (10 pts) Find the integral of $f(x,y,z) = x - z$ over the region bounded by $z = y^2$, $z = 1$, $z = x$ and $x = 0$.

7. (10 pts) Prove that [0,1] is a connected set.

8. (15 pts) Let $f(x)$ be an increasing function on [0,1]. Let $P_n = \{0, 1/n, 2/n, \ldots, 1\}$ be a partition on [0,1].
(a) Prove that for $i = 1, 2, \ldots, n$,

$$\sup\left\{f(x) : \frac{i-1}{n} \leq x \leq \frac{i}{n}\right\} = f\left(\frac{i}{n}\right) \quad \textit{and} \quad \inf\left\{f(x) : \frac{i-1}{n} \leq x \leq \frac{i}{n}\right\} = f\left(\frac{i-1}{n}\right).$$

(b) Let $U(f, P_n)$ be the upper Riemann sum and $L(f, P_n)$ be the lower Riemann sum. Prove that

$$0 \leq U(f, P_n) - L(f, P_n) = \frac{f(1) - f(0)}{n}.$$

(c) Use part (b) to prove that f is Riemann integrable on [0,1].

9. (15 pts) A function $f : \mathbf{R} \to \mathbf{R}$ is said to be continuous at a point a if given $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - a| < \delta$ then $f(x) - f(a)| < \epsilon$. Prove that $f : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ if and only if $f^{-1}(I)$ is open for every open interval I.