〈共5大題, 滿分爲100分〉

◇ To receive credit you MUST *SHOW YOUR WORK* ◇

Problem 1 (10 points) Find α, β and γ such that

$$\alpha\begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} + \beta\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \gamma\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Problem 2 (10 points) Find the kernel of the linear transformation $T(x,y,z) = (x+2y-3z; -2x+4z; 4y-2z; -2x-4y+6z)$. What is the nullity$(T)$ and rank(T)? Find a basis for the range of T.

Problem 3 (10 points) Let

$$A = \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & 10 \\ 0 & 0 & 4 \end{bmatrix}$$

Can we find an invertible matrix P such that $P^{-1}AP = \Lambda$ is a diagonal matrix? Find matrices P and Λ if exist.

Problem 4 (4+6=10 points) The following *Cayley-Hamilton Theorem* is well known for every $n \times n$ matrix A: *if* $p(\lambda)$ *is the characteristic equation of* A, *then* $P(A) = 0$.

(4a) If A^{-1} exists. How to find A^{-1} using the Cayley-Hamilton Theorem? (4 points)

(4b) Find A^{-1} using the method described in (4a) for the 3×3 matrix given in Problem 3. (6 points)

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Problem 5 ($6 \times 10 = 60$ points) Determine the following statements are **true** or **false**. Please give a proof if true or counterexample if false. *Answer without proof or counterexample will get no credit!*

(5a) (10 points) Let $T : V \to W$ be a linear transformation. Assume that $\dim V = n$, $\dim W = m$ and $n < m$. T is onto.

(5b) (10 points) Let A be an $n \times n$ singular matrix and

$$C \equiv (\text{adj } A)$$

denote the adjoint of A. The product AC is the zero matrix.

(5c) (5+5=10 points) Let A be an $m \times n$ matrix and A^T the transpose of A.

(5c-1) AA^T is invertible. (5 points)

(5c-2) $B = I + AA^T$ is nonsingular. (5 points)

(5d) (5+5=10 points) Assume that A is an $n \times n$ matrix having the following property:

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}| \quad \text{for } i = 1, 2, \cdots, n.$$

(5d-1) $\det A \neq 0$. (5 points)

(5d-2) A must be positive definite. (5 points)

(5e) (10 points) Let A denote a real $n \times n$ symmetric matrix. There exists a real symmetric matrix B such that $B^2 = A$.

(5f) (10 points) Assume that A and B are two real $n \times n$ matrices having the same rank and satisfying

$$A^2 = A \quad \text{and} \quad B^2 = B.$$

A and B must be similar.