There are 8 problems.

1. (15 pts) (a) Use the $\epsilon - \delta$ definition to prove that $1/x$ is continuous at each $x \in (0,1)$.
(b) Prove that $1/x$ is not uniformly continuous on (0,1).

2. (10 pts) Let $f(x,y)$ be a C^2 function on $\mathbf{R}^2$. Let $x = r\cos\theta$ and $y = r\sin\theta$. Prove that

$$f_{xx} + f_{yy} = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}.$$

3. (15 pts) (a) Suppose that f and f_n, n=1,2,3..., are defined on $(0,\infty)$, and are Riemann integrable on $[0,T]$. If $f_n \to f$ uniformly on $[0,T]$, prove that

$$\lim_{n\to\infty} \int_0^T f_n(x)\ dx = \int_0^T f(x)\ dx.$$

(b) If in addition, $f_n \to f$ uniformly on $[0,T]$ for each $T > 0$ and there is a function g defined on $(0,\infty)$, so that $|f_n| \le g$, and

$$\int_0^\infty g(x)\ dx < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\ dx = \int_0^\infty f(x)\ dx.$$

Hint :
$$\int_0^\infty |f_n(x) - f(x)|\ dx = \int_0^T |f_n(x) - f(x)|\ dx + \int_T^\infty |f_n(x) - f(x)|\ dx.$$

4. (10 pts) A point c is called a fixed point of a function f if $f(c) = c$.
(a) Prove that if f is continuous on [0,1] and $0 \le f(x) \le 1$ for all $x \in [0,1]$, then f has a fixed point.
(b) Prove that if f is differentiable on [0,1] and $f'(x) < 1$ for $x \in [0,1]$, then f has at most one fixed point in [0,1].
Hint: Consider the function $f(x) - x$.

5. (10 pts) Evaluate the surface integral

$$\int\int_S (x^2 + y^2 + z^2)\ d\sigma;$$

where S is the part of the plane $z = x + 2$ which lies inside the cylinder $x^2 + y^2 = 1$.

6. (10 pts) (a) Find the power series expansion for the function $f(x) = \ln(1-x)$.
(b) Find the interval of convergence of the power series obtained in (a).

7. (10 pts) A rectangular box without a top is to have a volume of 12 cubic feet. Find the dimensions of the box that will have minimum surface area.

8. (20 pts) For any $x = (x_1, x_2)$ in $\mathbf{R}^2$, we denote $\|x\| = \sqrt{x_1^2 + x_2^2}$.
Let A be a compact subset in $\mathbf{R}^2$ and $x_0 \notin A$.
(a) Prove that $\text{dist}(x_0, A) = \inf\{\|x - y\| : y \in A\} > 0$.
(b) Prove that there is a point $y_0 \in A$ so that $\|x_0 - y_0\| = \text{dist}(x_0, A)$.

Let B be a closed subset in $\mathbf{R}^2$ and $x_0 \notin B$.
(c) Prove that $\text{dist}(x_0, B) = \inf\{\|x - y\| : y \in B\} > 0$.
(d) Prove that there is a point $y_0 \in B$ so that $\|x_0 - y_0\| = \text{dist}(x_0, B)$.

Let A be a compact subset in $\mathbf{R}^2$ and B be a closed subset in $\mathbf{R}^2$. Suppose that A and B are disjoint.
(e) Prove that $\text{dist}(A, B) = \inf\{\|x - y\| : x \in A, \quad y \in B\} > 0$.
(f) Prove that there are points $x_0 \in A$ and $y_0 \in B$ so that $\|x_0 - y_0\| = \text{dist}(A, B)$.

END