Linear Algebra

* There are 5 problems with 100 points in this test.

** Show your work for partial credits.

1. Let A be an $m \times n$ matrix. L is a left inverse for A iff $LA = I_n$. R is a right inverse for A iff $AR = I_m$.
   (a) (10 pts.) Formulate a method for finding right inverse and left inverse and state the conditions for their existence.
   (b) (5 pts.) If $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \end{bmatrix}$, find two right inverse.

2. Let $p_0(t) = t^3$, $p_1(t) = t^2(t+2)$, $p_2(t) = 2t(t+1)(t-1)$, $p_3(t) = (3t+1)(t^2+2)$.
   (a) (5 pts.) Is $\beta = \{p_0, p_1, p_2, p_3\}$ a basis for the vector space P_3 of polynomials with degree less than or equal to 3? Explain your answer.
   (b) (5 pts.) Find the matrix M with Mb = a so that a and b are the coordinates for $p(t) \in P_3$ corresponding to the basis $\alpha = \{1, t, t^2, t^3\}$ and $\beta = \{p_0, p_1, p_2, p_3\}$ respectively.
   (c) (5 pts.) Find the coordinate of $t - 2t^2 + 3t^3$ corresponding to basis β.

3. Let $A = \begin{bmatrix} 1 & 1 & -1 & -3 \\ 2 & 0 & -2 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.
   (a) (10 pts.) Use two different methods to find a basis for $N(A)$.
   (b) (10 pts.) Use two different methods to find a basis for $R(A)$.
   (c) (5 pts.) Let the linear transformation $T: R^4 \to R^3$ to be $T(x) = Ax$. Is the transformation onto? Why? Give a geometric explanation for this linear transformation.

4. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 5 & -2 \\ 1 & -1 & 0 \end{bmatrix}$ and $V = \{x = (x_1, x_2, x_3)^T \in R^3 \mid x_1 - 2x_2 + x_3 = 0\}$.
   (a) (5 pts.) Define $T(x) = Ax$, show that $T: V \to V$.
   (b) (10 pts.) Find the eigenvectors and eigenvalues of T acting on V. Is A diagonalizable? Explain your answer.

5. (30 pts.) Prove or disprove the following statements.
   (a) If A and B are Hermitian matrices, then AB is always Hermitian.
   (b) If A is an $m \times n$ full row rank matrix, then $I - A^T(AA^T)^{-1}A = Z(Z^TZ)^{-1}Z^T$ for any Z which is a basis for $N(A)$.
   (c) If A and B are $n \times n$ matrices with the same eigenbasis, then AB=BA is diagonalizable.
   (d) If A is symmetric and the quadratic form $x^T Ax$ has only negative values for $x \neq 0$, then the eigenvalues of A are all negative.
   (e) If A and B are symmetric positive definite, then A+B is positive definite.