

1. Find the function $y = f(x)$, $x \in \mathbf{R}$ such that $y^2 x + y^2 \frac{dy}{dx} = x$ with initial condition $f(0) = 3$. (10 points)

2. Let $\mathbf{A} = \begin{pmatrix} -2 & 3 & 1 \\ 0 & 1 & 1 \\ -3 & 4 & 1 \end{pmatrix}$. Compute the following.

a. the eigenvalues and the corresponding eigenvectors of \mathbf{A} , (10 points)

b. rank of \mathbf{A} , (10 points)

c. \mathbf{A}^{25} , (10 points)

d. trace of \mathbf{A}^{50} : $\text{tr}(\mathbf{A}^{50})$. (10 points)

3. Show that the tangent line to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1. \quad (15 \text{ points})$$

4. Approximate

$$\int_0^1 e^{-x^2} dx$$

with an error less than $1/120$. (15 points)

5. Prove or disprove the following statements

a. $\int_1^\infty \frac{1}{x} dx$ diverges. (10 points)

b. $\int_1^\infty \frac{\sin^2 x}{x} dx$ diverges. (10 points)