Note: We encourage you to present your intuition or to describe your own viewpoint on the concerned subjects as widely or deeply as you can, eventhough your mathematical derivations are incomplete when solving the following problems:

1. (20%) Let Z denote the standard normal random variable. Derive the distribution of $Y = Z^2$ using two different methods and describe some features of the methods you choose. (you will get no more than 7 points if you use only one method to solve this problem).

2. (5%+5%+10%) Let $X_1, X_2, \cdots, X_n$ be the random samples of Normal distribution with $N(\alpha, \beta)$. Derive or prove that

   (a) the estimates $\hat{\alpha}$, $\hat{\beta}$ by MLE (Maximum Likelihood Estimator),

   (b) the estimates $\tilde{\alpha}$, $\tilde{\beta}$ by the method of moment, and

   (c) the (un)biasedness and (in)consistency of the above 4 estimates.

3. (20%) Let $\overline{X_1}$ and $\overline{X_2}$ be the sample means of two independent random samples, each of size n, from the respective distributions $N(\mu_1, 1.0)$ and $N(\mu_2, 1.0)$. Find the sample size n such that

$$Pr(\overline{X_1} - \overline{X_2} - 0.2 < \mu_1 - \mu_2 < \overline{X_1} + \overline{X_2} - 0.2) = 0.80$$

   where $z_{0.80} = 0.85$, $z_{0.90} = 1.28$ are the respective normal quantiles. Can you further discuss some related concepts of statistical confidence interval and its connection with the two-sided hypothesis testing ?

4. (20%) Let X denote a binomial distribution with $b(3, \theta)$. We shall use only one sample of X to test the simple hypothesis $H_0 : \theta = 0.5$ against the alternative simple hypothesis $H_a : \theta = 0.3$. How do you find the best critical region of size $\alpha = 0.125$? In addition, describe some related concepts on the statistical hypothesis testing based on your derivations of this problem.

5. (20%) A random sample $X_1, X_2, \cdots, X_n$ is from a distribution with the probability desity function

$$f(x; \theta) = e^{-(x-\theta)}, \quad \theta \le x < \infty,$$

   where $0 < \theta < \infty$. Find the MLE, $\hat{\theta}$, of θ and derive the probability density function of $\hat{\theta}$. In addition, you need to further explain the likelihood principle you have learned (it may be related to your derivations of this problem).