Questions 1 and 2 must be answered. Each possesses 20%.

1. Let S be a sample space and $E_i \cap E_j = \phi$ (i.e. E_i and E_j are disjoint events) for all $i \neq j$, $1 \leq i, j \leq n$ and $S = \bigcup_{i=1}^{n} E_i$.

(i) For some $P(E_i) > 0$, prove that $P(A \mid E_i) = P(A)$ if and only if $P(E_i \mid A) = P(E_i \mid A^c)$, where A^c is the complement of A.

(ii) If $p(A \cap E_i)=1/6$, $P(A^c \cap E_i)=1/2$, and $P(A \cap E_i^c)=1/6$, find the values of $P(A \mid E_i)$, $P(A)$, $P(E_i \mid A)$ and $P(E_i \mid A^c)$.

2. According to the following descriptions for random variable X from five distributions including Binomial, Poisson, Hypergeometric, Normal and Exponential, (a) give the probability density function, and
(b) give the name of the distribution.

(i) Description: X denotes the number of white balls selected in a sample of size n chosen randomly without replacement from an urn containing N balls, of which m are white and N-m are black.

(a)

(b)

(ii) Description: X denotes the weight of a bag of sugar bought from a store where it sold 1-pound sugar bags. The produced sugar bags have a standdard deviation equal to 0.1 pound.

(a)

(b)

(iii) Description: X denotes the number of times that earthquake occurs in a year, assuming that in the preceding years the average number of earthquakes that occured each year was 5.6.

(a)

(b)

(iv) Description: X denotes the waiting time until a shooter hit the target for the first time, assuming that the average amount of waiting time for a hit by the shooter is 5 minutes.

(a)

(b)

(v) Description: X denotes the number of wins when a person plays a game n times, assuming that the person has probability of winning the game each time equal to p.

(a)

(b)

Answer three questions among Questions 3 to 7 in what follows.

3. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. (20%)

(i) Express what is The Central Limit Theorem.

(ii) Express what is The Weak Law of Large Numbers.

(iii) Prove The Weak Law of Large Numbers using The Central Limit Theorem.

4. Suppose that earthquakes occur in the eastern portion of Taiwan with $\lambda = 2$ and with 1 week as the unit of time. (That is, earthquakes occur at a rate of 2 per week.) (20%)

(i) Find the probability that at least 3 earthquakes occur during the next 2 weeks.

(ii) Find the probability distribution of the time, starting from now, until the next earthquake.

5. For a nonnegative integer-valued random variable N, (20%)

(i) Show that $E[N] = \sum_{i=1}^{\infty} P\{N \geq i\}$

(ii) Show that $\sum_{i=0}^{\infty} i\, P\{N > i\} = \frac{1}{2}(E[N^2] - E[N])$

Hint: $\sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P\{N=k\}$ and $\sum_{i=0}^{\infty} iP\{N>i\} = \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P\{N=k\}$.

6. One thousand independent rolls of a fair die will be made. (20%)

(i) Compute an approximation to the probability that number 6 will appear between 150 and 200 times.

(ii) If number 6 appears exactly 200 times, find the probability that number 5 will appear less than 150 times.

7. Suppose that the joint density of X and Y is given by (20%)

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty, \ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(i) Are X and Y independent? Why or why not?

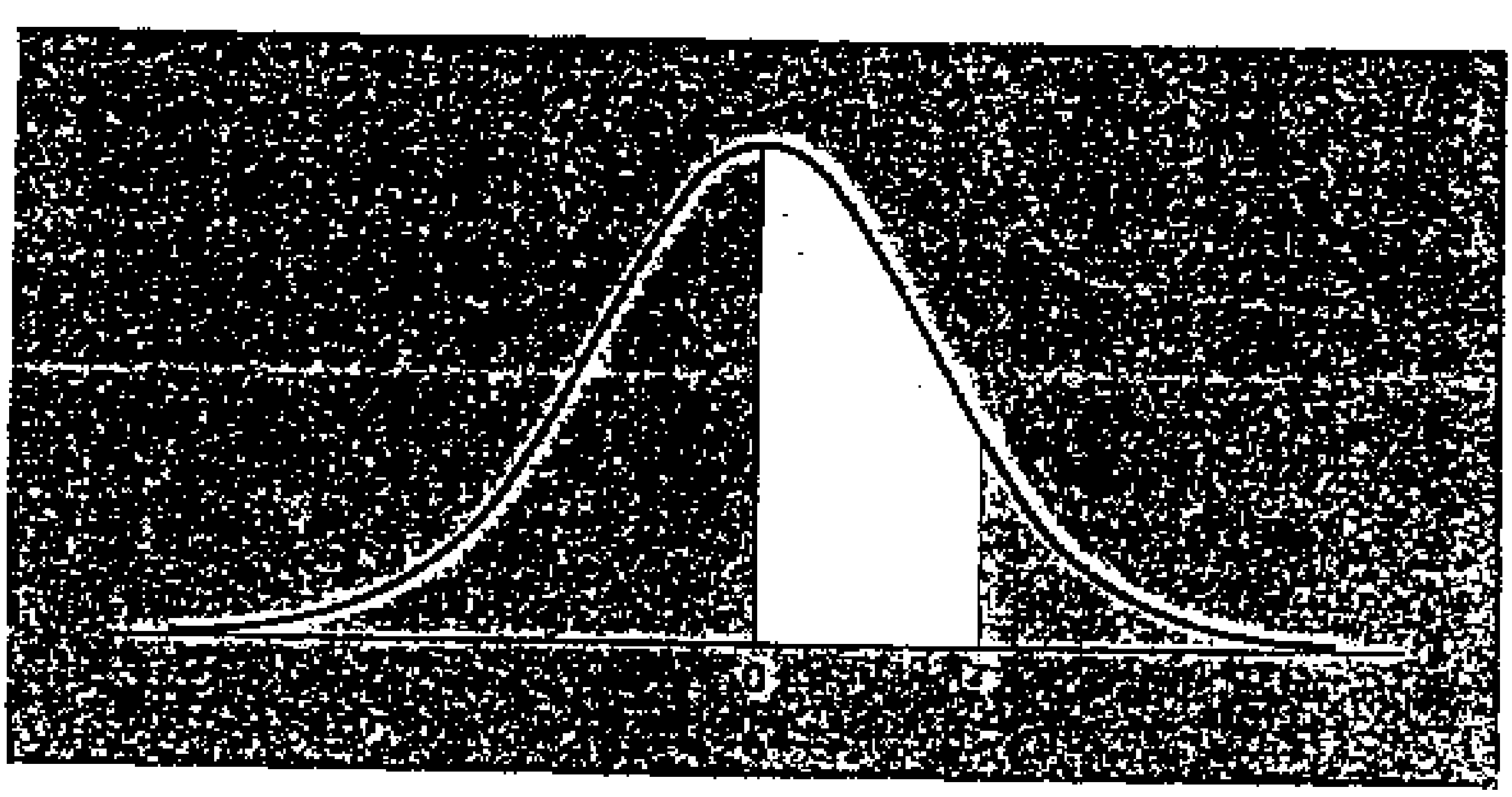
(ii) Find the conditional density function of X, given that Y=y.

(iii) Find $P\{X > 1 \mid Y > 1\}$.

TABLE II NORMAL-CURVE AREAS

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Also, for z = 4.0, 5.0, and 6.0, the areas are 0.49997, 0.4999997, and 0.499999999.



The entries in Table II are the probabilities that a random variable having the standard normal distribution will take on a value between 0 and z. They are given by the area of the white region under the curve in the figure shown above.