Note: We encourage you to present your intuition or to describe your own viewpoint on the concerned subjects as widely or deeply as you can, eventhough your mathematical derivations are incomplete when solving the following problems:

1. (20%) Let the simple linear regression model be

$$y_i = \alpha + \beta x_i + \epsilon_i \, , i = 1, \cdots, n,$$

   where ϵ_i are i.i.d. from the normal distribution with the mean zero and the variance σ^2. Find the maximum likelihood estimators of α and β, respectively. Moreover, Derive the respective distributions of both estimators and the correlation between them.

2. (20%) A random sample $X_1, X_2, \cdots, X_n$ is from the Uniform distribution $U(0, \theta) \, , 0 < \theta < \infty$. Find the MLE, $\hat{\theta}$, of θ and describe the probability density function of $\hat{\theta}$. Is this MLE unbiased ? In addition, you need to further explain the likelihood principle you have learned (it may be related to your derivations of this problem).

3. (20%) Given two correlated random variables X, Y, we can construct another random variable $E(Y|X)$. Derive some relationships among them with

   (a) the expectations and variances of Y and $E(Y|X)$,

   (b) the correlation between X and $Y - E(Y|X)$.

   Furthermore, present some applications with these results as you know.

4. (40%) Let $X_1, X_2, \cdots, X_n$ be the random samples from the Bernoulli distribution with $B(1, \theta)$. Derive or formulate

   (a) the point estimators $\hat{\theta}$ by the method of moment, the MLE (Maximum Likelihood Estimator), respectively,

   (b) the (un)biasedness and (in)consistency of the above estimators,

   (c) the connections with the Law of Large Number and the Central Limit Theorem based on the above derivations,

   (d) an interval estimation with the confidence coefficient $1 - \alpha$,

   (e) a test statistic for checking $\theta = 0.5$ or not.

   Moreover, describe some related properties you have learned.