(10%) 1. The Gamma Function is defined by

\[ \Gamma(x) = \int_0^\infty x^{a-1} e^{-x} \, dx, \quad a > 0. \]

(a) Show that \( \Gamma(a + 1) = a \Gamma(a) \).
(b) Find \( \Gamma\left(\frac{1}{2}\right) \).

(10%) 2. Evaluate \( \int_0^2 \int_{x^2}^{x^4} \sqrt{x \sin x} \, ds \, dy \).

(10%) 3. Let \( R \) be the region bounded by the square with vertices \((0,1), (1,2), (2,1)\) and \((1,0)\). Evaluate the integral

\[ \iint_{R} (x+y)^2 \sin(x-y) \, dA. \]

(10%) 4. Find the Taylor series for the function

\[ f(x) = e^x \sin x. \]

(10%) 5. Evaluate \( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \).

(10%) 6. Consider the linear transformation defined by

\[ u(x) = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_m - \bar{x} \end{bmatrix}, \]

for all \( x \in \mathbb{R}^m \), where \( \bar{x} = \frac{1}{m} \sum x_i \).
(a) Find the matrix \( A \) for which \( u(x) = Ax \).
(b) Determine the dimension of the range and null spaces.
(30%) 7. Consider the $m \times m$ matrix $A = \alpha I_m + \beta 1_m 1'_m$, where $\alpha$ and $\beta$ are scalars.

(a) Find the eigenvalues and eigenvectors of $A$.

(b) Determine the eigenspaces and associated eigenprojections of $A$.

(c) For which values of $\alpha$ and $\beta$ will $A$ be nonsingular?

(d) Using (a), show that when $A$ is nonsingular, then

$$A^{-1} = \alpha^{-1} I_m - \frac{\beta}{\alpha(\alpha + m\beta)} 1_m 1'_m.$$ 

(e) Show that the determinant of $A$ is $\alpha^{m-1}(\alpha + m\beta)$.

(f) Determine values of $\alpha$ and $\beta$ so that $A$ is an idempotent matrix.

(10%) 8. Find a $3 \times 2$ matrix $T$ such that $TT' = A$, where

$$A = \begin{bmatrix} 5 & 4 & 0 \\ 4 & 5 & 3 \\ 0 & 3 & 5 \end{bmatrix}.$$