Total 7 problems

(15%) 1. Let \( T(x, y, z) = 20 + 2x + 2y + z^2 \) represent the temperature at each point on the sphere \( x^2 + y^2 + z^2 = 11 \). Use Lagrange multipliers to find the extreme temperatures on the curve formed by the intersection of the plane \( x + y + z = 3 \) and the sphere.

(15%) 2. Let \( R = \{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \} \). Evaluate the integral

\[
\int \int \int_R xyz\, dx\, dy\, dz.
\]

3.

(10%)(a) Find the series representation of \( e^{-x^2} \) and determine its interval of convergence.

(5%)(b) Use a power series to approximate \( \int_0^1 e^{-x^2} \, dx \) with an error of less than 0.01.

(10%) 4. Find the derivative of the function

\[
P(z) = \frac{[x^2]}{1 + x^2}
\]

whenever it exists. (\([x]\) is the greatest integer function)

(15%) 5. Find an orthogonal matrix \( P \) and a diagonal matrix \( D \) such that \( P^{-1}AP = D \)

where \( A = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} \).

(10%) 6. A matrix \( P \) is idempotent if \( P^2 = P \). If \( P \) is symmetric, then \( P \) is idempotent and of rank \( r \) if and only if it has \( r \) eigenvalues equal to unity and \( n - r \) eigenvalues equal to zero.

(20%) 7. Let \( T \) be the linear operator on \( \mathbb{R}^2 \) define by

\[
T(x_1, x_2) = (-x_2, x_1).
\]

(a) What is the matrix of \( T \) in the standard ordered basis for \( \mathbb{R}^2 \)?

(b) What is the matrix of \( T \) in the ordered basis \( B = \{(1, 2), (1, -1)\} \)?

(c) Prove that for every real number \( c \) the operator \((T - cf)\) is invertible.

(d) Prove that if \( B \) is any ordered basis for \( \mathbb{R}^2 \) and \([T]_B = A\), then \( a_{12}a_{21} \neq 0 \).