國立中正大學九十二學年度碩士班招生考試試題

系所別：統計科學研究所　　　　科　目：機率與統計

Total 5 Problems

注意: 下列題目的作答、推導、證明等, 把握住重點, 過程力求簡短、扼要。

1. (5%) + (10%)
   Suppose that $X_1, X_2, \cdots, X_n$ are the random samples from Normal population with $N(\mu, \sigma^2)$, i.e., the probability density function is

$$f(x_i;\ \mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma}\ \exp\{-\frac{1}{2}(\frac{x_i-\mu}{\sigma})^2\}\ ,\ -\infty < x_i < \infty\ ,\ i = 1, 2, \cdots, n.$$

   (a) Derive the distribution of $Z = (\frac{X_1-\mu}{\sigma})^2$ using *two different* methods.

   (b) Derive the estimator of σ^2 by the method of moments, denoted by W.
   On the other hand, find its unbiased estimator, denoted by U. Moreover, compare their properties by the variance and the mean squared error, respectively.

2. (5%) + (5%) + (5%) + (5%)
   Let $X_1, X_2, \cdots, X_n$ be the independent exponential random variables with rates $\lambda_1, \cdots, \lambda_n$, respectively, i.e., the density function is $f(x_i;\ \lambda_i) = \lambda_i\ \exp(\lambda_i\, x_i)$, $x_i > 0, i = 1, \cdots, n$.
   Let $Y_1 = min\{X_1, X_2, \cdots, X_n\}$.

   (a) Derive the probability of the event of $X_1 < X_2$.

   (b) Derive the distribution of the order statistic Y_1.

   (c) Find the probability of the event of $Y_1 = X_i$ for some $1 \le i \le n$.

   (d) Find and prove the distribution of the random variable $W = X_1 + X_2 + \cdots + X_n$ when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$.

3. (5%) + (5%) + (5%) + (5%) + (5%)
   Let $Y_1, Y_2, \cdots, Y_k$ be the outcomes of Bernoulli trial, i.e., they are the random samples from Bernoulli distribution with parameter θ.

   (a) Write down its likelihood function and derive the Maximum Likelihood Estimator (MLE) of θ, denoted by $\hat{\theta}$.

   (b) Derive the moment generating function of $\hat{\theta}$.

   (c) Derive the estimator of θ by the method of moments, denoted by $\tilde{\theta}$, and find its limiting distribution of the derived estimator $\tilde{\theta}$.

   (d) Find the probability mass function of $k\hat{\theta}$ and derive its Poisson approximation when k is large enough and θ is small enough.

   (e) Find and prove the unbiased estimator of $\mathrm{Var}(\hat{\theta})$.

4. (5%) + (5%) + (5%) + (5%) + (10%)
   Suppose that $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ are the random samples from two independent Normal populations with $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$, respectively.
   Let $\theta = \mu_X - \mu_Y$ and $\delta = \sigma_X^2/\sigma_Y^2$.

   (a) Derive the suitable estimators of θ, δ, denoted by $\hat{\theta}$, $\hat{\delta}$, respectively.

   (b) Find the distribution of your derived estimators $\hat{\theta}$ and $\hat{\delta}$ and described their properties, respectively.

   (c) Find the critical region of size $\alpha = 0.05$ for testing $H_0 : \theta = 0$ against $H_1 : \theta < 0$ when the sample sizes are large.

   (d) Find the critical region of size $\alpha = 0.01$ for testing $H_0 : \delta = 1$ against $H_1 : \delta > 1$.

   (e) Describe and discuss the $100(1-\alpha)\%$ confidence interval of $\hat{\theta}$ when the sample sizes are small.

5. (5%) + (5%)
   Let $X_1, X_2, \cdots, X_n$ be the random sample from Poisson(λ), i.e., the probability mass function is

$$f(x_i; \lambda) = e^{-\lambda}\,\frac{\lambda^{x_i}}{x_i!}\ ,\ x_i = 0, 1, 2, \cdots\ ;\ i = 1, 2, \cdots, n.$$

   (a) Find two *different* unbiased estimators of λ, denoted by U, W, respectively and show their unbiasedness.

   (b) Find and prove the *best unbiased* estimator of λ.