國立中正大學九十三學年度碩士班招生考試試題

系所別：統計科學研究所　　　　科目：基礎數學

共七題 總分100分　　　　

This exam has 7 questions, for a total of 100 points.

1. Let

$$f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$$

   (a) (5 points) Find $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$.

   (b) (5 points) Is $f(x,y)$ continuous at $(0,0)$? Why or why not?

2. (10 points) Find

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}.$$

3. (15 points) Suppose $g(x)$ is a C^∞ function defined on $[a,b]$ with $g^{(n+1)}(x)=0$ for all $x\in(a,b)$, where $a<b$. Show that $g(x)$ has at most n real roots for $n\geq 1$.

4. (15 points) Let $f(x)$ be a continuous function on $[c,d]$, where $c<d$ and $f(x)\geq 0$. If $\int_c^d f(x)\,dx=1$, show that for any nonzero constant k

$$(\int_c^d f(x)\cos kx\,dx)^2+(\int_c^d f(x)\sin kx\,dx)^2<1.$$

5. Let $\mathbf{x}'=(x_1,x_2,\ldots,x_n)$ be a real vector where $n\geq 2$. Define

$$s^2=\frac{1}{n-1}[\sum_{i=1}^n x_i^2-(\sum_{i=1}^n x_i)^2/n].$$

   (a) (10 points) Find an $n\times n$ matrix $\mathbf{P}$ such that $\mathbf{x}'\mathbf{P}\mathbf{x}=(n-1)s^2$.

   (b) (10 points) Find the determinant of $\mathbf{P}$.

6. (15 points) Let $\mathbf{M}$ be a $t\times u$ matrix. Denote $\mathcal{S}(\mathbf{M})$ be the linear subspace of $\mathbb{R}^t$ spanned by the columns of the matrix $\mathbf{M}$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two $n\times 1$ nonzero real vectors. Show that $\mathcal{S}(\mathbf{u}\mathbf{v}')=\mathcal{S}(\mathbf{u})$.

7. (15 points) Find the minimum and maximum values of the ratio $\mathbf{u}'\mathbf{A}\mathbf{u}/\mathbf{u}'\mathbf{u}$ for any nonzero real vector $\mathbf{u}'=(u_1,u_2,u_3)$, if

$$\mathbf{A}=\begin{bmatrix}13 & -4 & 2\\ -4 & 13 & -2\\ 2 & -2 & 10\end{bmatrix}.$$