共七題，總分100分

1. (10%) A group of N people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects a hat. Let X be the number of people that select their own hat. Find $E(X)$ and $\text{var}(X)$.

2. (15%) Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2 respectively.

   (a) (10%) Find the distribution of $X+Y$.

   (b) (5%) Compute the conditional distribution of X, given that $X+Y=n$.

   Note: You must write down the computing process in details.

3. (15%) A coin is selected from a box and flipped three times. Let A be the event of {Head,Tail,Head}. Suppose that the probability that the selected coin occurs a head is a *random variable* Y, where Y is assumed to be uniformly distributed on $[0,1]$.

   (a) (10%) Compute $P(A)$ and the conditional probability density function $f_{Y|A}(y|A)$.

   (b) (5%) For what value of y does this conditional density have a maximum value?

4. (15%) Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population $N(0,1)$.

   (a) (5%) Show that

   $$V_n = \frac{X_1^2 + X^2 + \ldots + X_n^2}{n}$$

   converges in probability to 1.

   (b) (10%) Find the limiting distribution of

   $$W_n = \sqrt{n}\,\frac{X_1 + X_2 + \ldots + X_n}{X_1^2 + X_2^2 + \ldots + X_n^2}.$$

5. (10%) Let $X_1, \ldots, X_n$ be a random sample of size n from a distribution with the p.d.f.

   $$f(x;\theta) = \frac{3\theta^3}{x^4} \qquad \text{for } x \geq \theta,$$

   where $\theta > 0$ is an unknown parameter. Find the maximum likelihood estimate $\hat{\theta}$ and a one-dimensional sufficient statistic.

6. (15%) Let X be a random variable whose p.d.f. is given by

   $$H_0 : f(x) = \begin{cases} 2x, & 0 < x < 0.5 \\ 2-x, & 0.5 \leq x < 1 \end{cases} \quad v.s. \quad H_1 : f(x) = 1, \text{ if } 0 < x < 1.$$

(a) (10%) State the Neyman-Pearson Lemma. On the basis of one observation, find a most powerful test of size α, assuming that $0 < \alpha < 0.5$.

(b) (5%) Find the power of your test.

7. (20%) Suppose that $X_1, X_2, \ldots, X_n$ are a random sample from a uniform population $U(0, \theta)$. Define $M_n = \max\{X_1, \ldots, X_n\}$.

(a) (10%) Find the p.d.f. for M_n and show that $E_\theta(M_n - \theta)^2 = 2\theta^2/(n+1)(n+2)$.

(b) (10%) Use the Chebyshev inequality to show that an approximate $(1-\alpha)$ confidence interval for θ is

$$\left(M_n, M_n\left(1 + \frac{1}{n}\sqrt{\frac{2}{\alpha}}\right)\right).$$