This exam has 7 questions, for a total of 100 points.

1. (a) (5 points) Let $f(x,y) = x^2 + 2xy + 3y^2$, find the Taylor's expansion of $f(x+2, y+1)$ at point $(2,1)$.
   (b) (5 points) Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

2. (10 points) Evaluate
$$\lim_{x\to 5}\left(\frac{x}{x-5}\int_5^x \frac{e^t}{t}\,dt\right).$$

3. (10 points) Suppose function $f(x)$ and $g(x)$ are continuous on $[a,b]$ and they are differentiable on (a,b) where a and b are two real constants. Show that we can find a value c such that $\begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & g(a) \\ f'(c) & g'(c) \end{vmatrix}$.

4. (a) (5 points) Prove that if f is a continuous function on $\mathbb{R}$, then for any constant a,
$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$
   (b) (10 points) Given a positive integer n, use part (a) to evaluate
$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx.$$

5. (15 points) Find the values of a, b, and c such that the polynomial function $f(x) = ax^2 + bx + c$ has a minimum sum of squared errors for the points $(-2,0), (-1,1), (1,2), (2,3)$.

6. Given a matrix $\mathbf{\Sigma} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$ where $0 < \rho < 1$.
   (a) (10 points) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{\Sigma}$.
   (b) (15 points) If $\rho = 1/2$, find a matrix $\mathbf{V}$ such that $\mathbf{V}^2 = \mathbf{\Sigma}^{-1}$ and evaluate $\det(V)$.

7. (15 points) Given an $m \times n$ real matrix $\mathbf{A}$ and an $m \times 1$ real vector b. Prove that the system of linear equation $\mathbf{AX} = b$ has solution(s) if and only if b is in the column space of $\mathbf{A}$.