1. (10%) A box contain 4 red balls and 6 blue balls. Two balls are drawn at random and, without seeing their colors, are thrown out. If a third ball is drawn randomly and found to be a red ball, what's the probability that the previous two balls are blue?

2. (10%) Suppose that $X$ has density function $f(x) = 6x(1-x), 0 < x < 1$, and $Y$ is uniformly distributed on $[0, X]$. Find the probability density function of $Y$. Given $Y = 0.5$, what's the probability that $X < 1$?

3. (12%) Let $X$ and $Y$ be independent exponential random variables with $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$. Define $Z = \min\{X, Y\}$ and

$$W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y. \end{cases}$$

Find the joint distribution of $Z$ and $W$, and show that $Z$ and $W$ are independent.

4. (20%) Sampling distribution

   (a) (10%) Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Show that $\bar{X}$ and $(X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})$ are independent.

   (b) (10%) The moment generating function of $\chi_n^2$ is known as $M(s) = (1-2s)^{-n/2}$ (you don't need to show this). Define

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \quad \text{and} \quad T_n = \frac{\bar{X} - \mu}{S_n}.$$

   Use Chebychev's inequality to find the limiting distribution of $S_n^2$. Also find the relationship between the limiting distribution of $T_n$ and $N(0,1)$. Please verify your answer clearly.

5. (10%) Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ populations, respectively. Derive a two-sided $(1-\alpha)$ confidence interval for $(\mu_1 - \mu_2)$. Explain your steps in all the details. You may need to use Problem 4(a).

6. (15%) We say that $\{p(x, \theta) : \theta \in \Theta\}$ is monotonic likelihood ratio (MLR) in statistics $T(x)$ if for $\theta_1 < \theta_2$, $p(x, \theta_2)/p(x, \theta_1)$ is nondecreasing in $T(x)$.

   (a) (9%) For testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_1$, show that $T(x)$ can construct a uniformly most powerful (UMP) test of size $\alpha$.

(b) (6%) Let $X_1, X_2, \ldots, X_n$ be a random sample with the p.m.f.

$$P_\theta\{X = x\} = \frac{1}{\theta}, x = 1, \ldots, \theta, \tag{1}$$

where $\theta \in \Theta = \{1, 2, \ldots,\}$. Find $T(x)$ such that the joint p.m.f. $P_\theta(x)$ is a MLR in $T(x)$ and then construct a UMP test for testing $H_0 : \theta \le \theta_0$ against $H_1 : \theta > \theta_0$.

7. (23%) Uniformly most powerful unbiased estimate (UMVUE)

(a) (5%) State the definition of complete sufficient statistics (CSS).

(b) (10%) State and prove the Lehmann-Scheffe theorem.

(c) (8%) Let $X \sim P_\theta(x)$, where $P_\theta(x) = \frac{1}{\theta}$, $x = 1, \ldots, \theta$ (the same distribution appearing in eq (1).) Find the CSS and UMVUE for $\theta$. Please verify your answer clearly.