Calculus

Part I. No partial credit will be given in this part. Do not need to show all your work. Only the final result will be needed.

(1) (6 points) Find all points on the graph of the function \( f(x) = 2 \sin x + \sin^2 x \), \( 0 \leq x < 2\pi \), at which the tangent line is horizontal.

(2) (6 points) A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a constant speed of 15 ft/s. When he passes under the balloon it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 seconds later?

(3) (6 points) Find the local maximum and minimum values of the function \( f(x) = x^{1/3}(x + 3)^{2/3} \).

(4) (6 points) Evaluate \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n + \left(i^2/n\right)} \).

(5) (6 points) The marginal cost of manufacturing \( x \) yards of a certain fabric is \( C'(x) = 3 - 0.01x + 0.000006x^2 \) (in dollars per yard). Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

(6) (6 points) If \( x \sin \pi x = \int_{0}^{x^2} f(t) \, dt \), where \( f \) is a continuous function, find \( f(4) \).

(7) (6 points) Suppose \( f \) is an increasing function, \( f'(x) > 0 \), \( f(0) = 0 \), \( f(1) = 1 \), and \( \int_{0}^{1} f(x) \, dx = \frac{1}{3} \). Find the value of the integral \( \int_{0}^{1} f^{-1}(y) \, dy \), where \( f^{-1} \) is the inverse function of \( f \).
(8) (6 points) Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by \( y = x - x^2 \) and \( y = 0 \) about the line \( x = 2 \).

(9) (18 points) Evaluate the integral.
\[
\begin{align*}
    (a) & \int_{0}^{\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} \, dx \\
    (b) & \int_{0}^{3} \frac{1}{x^2 - x - 2} \, dx \\
    (c) & \int x^5 e^{-x^3} \, dx
\end{align*}
\]

(10) (6 points) If \( f(x, y) = xe^y \), find the rate of change of \( f \) at the point \( P(2, 0) \) in the direction toward the point \( Q(\frac{1}{2}, 2) \). In which direction does \( f \) have the maximum rate of change? What is the maximum rate of change?

(11) (6 points) Evaluate the double integral \( \int \int_D y^3 \, dA \), where \( D \) is the triangular region with vertices \((0, 2), (1, 1), \) and \((3, 2)\).

**Part II.** Partial credits will be given in this part. Show all your work to get credits.

(12) (8 points) Find the interval of convergence of the series \( \sum_{n=1}^{\infty} n^3 x^n \) and find its sum.

(13) (7 points) Show that when Laplace's equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \) is written in cylindrical coordinates \((r, \theta, z)\), it becomes \( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \).

(14) (7 points) Evaluate the line integral \( \int_C yz \, dx + xz \, dy + xy \, dz \), where \( C \) consists of line segments from \((0, 0, 0)\) to \((2, 0, 0)\), from \((2, 0, 0)\) to \((1, 3, -1)\), and from \((1, 3, -1)\) to \((1, 3, 0)\).