

Please answer ALL of the following 7 questions.

1. (20%) Let A be a square matrix over field \mathbb{F} . Show that A is invertible if and only if $\det(A) \neq 0$.
2. (10%) Give an example that a subset in \mathbb{R}^2 is NOT a subspace of \mathbb{R}^2 . Prove your example.
3. (10%) Applying the Gram-Schmidt Process to the basis $B = \{1, x, x^2\}$ in P_2 with the inner product $\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$.
4. (15%) Let A be a $m \times n$ matrix over field \mathbb{R} . Define $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$; $R(A) = \{Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n\}$. Prove that $N(A) = R(A^T)^\perp$.
5. (15%) Find a matrix P that orthogonally diagonalizes the matrix

$$\begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}.$$

6. (15%) Let A be a $n \times n$ matrix with characteristic polynomial

$$f = (x - a_1)^{b_1} \cdots (x - a_k)^{b_k}.$$

Show that $\text{Trace}(A) = a_1 b_1 + \cdots + a_k b_k$.

7. (15%) Let A be a 5×5 complex-valued matrix with characteristic polynomial $f = (x - 2)^3(x + 7)^2$ and minimal polynomial $p = (x - 2)^2(x + 7)$. Find the Jordan form of A .

This exam has 7 questions, for a total of 100 points.

1. (10 points) Evaluate

$$\int_0^{\infty} e^{-2x^2} dx.$$

2. (a) (8 points) Suppose

$$f(x) = \int_x^{x^2} t / \log(t^2 + 1) dt,$$

find $f'(2)$.

- (b) (10 points) Find the function $y = f(x)$, $x \in \mathbb{R}$ such that

$$y^2 x \frac{dy}{dx} - x^2 + 1 = 0$$

with initial condition $f(1) = 0$.

3. Suppose $\sum_{n=1}^{\infty} |a_n|$ is finite. Determine whether the following series is convergent or not and show your reasons.

(a) (10 points) $\sum_{n=1}^{\infty} |a_n|^{1/2}$.

(b) (10 points) $\sum_{n=1}^{\infty} a_n$.

4. (12 points) Denote

$$h(\alpha) = \begin{cases} \frac{1}{\lambda(\lambda+1)} \{ \alpha^{1-\lambda} + (1-\alpha)^{1-\lambda} - 1 \} & \text{if } \lambda \neq 0 \\ \frac{1}{2} \{ -\alpha \ln \alpha - (1-\alpha) \ln(1-\alpha) \} & \text{if } \lambda = 0, \end{cases}$$

where $-1 < \lambda$ is a constant and $0.1 < \alpha < 1$. Find the maximum of h .

5. (10 points) Given $\mathbf{v}_1 = (-1, -2, -2)$, $\mathbf{v}_2 = (0, 1, 4)$, $\mathbf{v}_3 = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$ in \mathbb{R}^3 . Write \mathbf{w} as a linear combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 .

6. Let \mathbf{A} , \mathbf{B} , and \mathbf{C} be $n \times n$ matrices. Determine whether each of the following statements is true or false. If the statement is true, prove it. Otherwise, give a counterexample.

(a) (5 points) If $\mathbf{AC} = \mathbf{BC}$, then $\mathbf{A} = \mathbf{B}$.

(b) (5 points) If \mathbf{A} and \mathbf{B} are invertible, then \mathbf{AB} is invertible.

7. Suppose $\mathbf{A} = \mathbf{I} - \frac{1}{n} \mathbf{J}' \mathbf{J}$ where \mathbf{I} is an $n \times n$ identity matrix and $\mathbf{J} = (1, 1, \dots, 1)$ is a $1 \times n$ vector, $n > 1$.

(a) (10 points) Show that \mathbf{A} is idempotent.

(b) (10 points) Find the eigenvalues and the corresponding eigenvectors of \mathbf{A} .

1. (10%) Let X_i be the amount of payment per insurance case which follows an exponential distribution with $E(X_i) = 4$. That is,

$$f_{X_i}(x) = \begin{cases} \frac{1}{4}e^{-\frac{1}{4}x} & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Let N be the number of insurance case and follow a Poisson distribution with $E(N) = 10$. Assume that X_i and N are independent.

- (i) (5%) Find $E(X_1 + X_2 + \cdots + X_N)$.
(ii) (5%) Find $Var(X_1 + X_2 + \cdots + X_N)$.
2. (10%) Let X_1, X_2, \dots, X_n denote a random sample from a normal distribution $N(0, \sigma^2)$. Now $n = 4$, let $U = (X_1, X_2, X_3, X_4)$, and $Y = \|U\| = \sqrt{X_1^2 + X_2^2 + X_3^2 + X_4^2}$.

- (i) (3%) Assume $W \sim \chi_{(4)}^2$, find a c such that $F_Y(y) = Pr(W \leq c)$.
(ii) (7%) Based on (i), find the pdf of Y . Hint: Let $V \sim Gamma(\alpha, \beta)$, then

$$f_V(v) = \begin{cases} \frac{v^{\alpha-1}e^{-v/\beta}}{\Gamma(\alpha)\beta^\alpha} & 0 < v < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

3. (10%) There is a quick way to generate a simulated data from $N(0, 1)$. The method is described as follows. "First, we use computer to generate 12 observations (say X_1, X_2, \dots, X_{12}) from $U(0, 1)$. Next, sum of these random numbers and subtract 6 and denote it by R . Finally, this obtained value seems to come from $N(0, 1)$."

- (i) (5%) Explain what is the theory behind this method in details. Specifically, why does R have an approximate distribution of $N(0, 1)$.
(ii) (5%) Is the exact distribution of R really normal? Give the reasoning for your answer.

4. (15%)

- (i) (3%) Give the definition of the MLE.
(ii) (5%) Let $X \sim Bin(1, p)$, $p \in [\frac{1}{5}, \frac{4}{5}]$. Find the MLE of p .
(iii) (7%) Consider a random sample from $Gamma(1, \beta)$. It is required to find the MLE of β in the following manner. A sample of size n is taken, and it is known only that k , $0 \leq k \leq n$, of these observations are $\leq M$, where M is a fixed positive number. Let $p = P(X_i \leq M) = 1 - e^{-M/\beta}$. Find the MLE of β .

5. (20%) Let X_1, X_2, \dots, X_n be i.i.d with pdf

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{1}{\theta}x} & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

(i) (4%) Find a uniformly minimum variance unbiased estimator (UMVUE) of θ .

(ii) (6%) Find a uniformly most powerful critical region of size $\alpha = 0.05$ and $n = 2$ for testing $H_0 : \theta = 2$ against $H_a : \theta > 2$. Show the details. ($\chi_{(2)}^2 = 5.991, \chi_{(3)}^2 = 7.815, \chi_{(4)}^2 = 9.488, \chi_{(5)}^2 = 11.071, \chi_{(6)}^2 = 12.592$)

(iii) (6%) When $n = 3$ and let $Z_1 = X_1 + X_2 + X_3, Z_2 = X_2 + X_3, Z_3 = X_3$, find the joint pdf of Z_1, Z_2, Z_3 .

(iv) (4%) Find $E\left(\frac{Z_1}{3} | z_3\right)$.

6. (20%) For the simple linear regression model, $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $i = 1, \dots, n$ and ϵ_i are i.i.d $N(0, \sigma^2)$.

(i) (6%) To derive the MLE (Maximum Likelihood Estimator) for β_0, β_1 and σ^2 .

(ii) (6%) Are $\hat{\beta}_0$ and $\hat{\sigma}^2$ unbiased estimators of β_0 and σ^2 , respectively? Show the details.

(iii) (4%) Find $Var(\hat{\beta}_1)$.

(iv) (4%) Show the sum of squares regression (SSReg), $SSReg = \hat{\beta}_1^2 S_{xx}$, where $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2$.

7. (15%) 有一種拍賣方式，參加者每人將自己的出價金額寫在紙上，以記名的方式投入一黑箱，在所有人皆放入自己的紙條後，最後再開箱唱名，由出價最高者得標，但是卻以次高價付款，身為拍賣會顧問的你，想要了解得標者的付款金額的隨機現象，因此你做了以下的數學假設：

假設 X_i 為每人出價金額 (單位：千萬)， Y 為得標者付款金額，並假設每人出價金額彼此獨立且來自於相同的均勻分配 $U(0, 1)$ 。今共有 15 人參與投標。

(i) (3%) 請問得標者的付款金額可以用統計學上的哪個統計量做描述？

(ii) (7%) 請求出 Y 的機率密度函數，並說明此為哪個著名的機率分配。

(iii) (5%) 請求出 Y 的期望值。