國立中正大學102學年度碩士班招生考試試題

系所別：數學系、數學系應用數學　　　　科目：高等微積分

第1節　　　　

1. [15 points] Evaluate the integral
$$\int_0^\infty e^{-8x^2}\,dx.$$

2. [20 points] Define
$$f(x,y)=\begin{cases}\frac{x^3-xy^2}{x^2+y^2} & (x,y)\neq(0,0),\\ 0 & (x,y)=(0,0).\end{cases}$$

   (a) [7 points] Show that f is continuous at $(0,0)$.

   (b) [6 points] Show that f has directional derivatives in all directions at $(0,0)$.

   (c) [7 points] Show that f is not differentiable at $(0,0)$.

3. [15 points]

   (a) [7 points] Let $f:[a,b]\to\mathbb{R}$ be a differentiable function and and $f'(a)<\alpha<f'(b)$. Show that there exists $c\in[a,b]$ such that $f'(c)=\alpha$.

   (b) [8 points] Let $g:[a,b]\to\mathbb{R}$ be a differentiable function. Use part (a) to show that if $g'(c-)$ and $g'(c+)$ exist for some $c\in[a,b]$, then $g'(c-)=g'(c+)$.

4. [20 points] Let f, g be two continuous functions defined on $[a,b]$ and $|g(x)|>0$ for $x\in[a,b]$. Suppose that f_n and g_n converge uniformly on $[a,b]$ to f and g, respectively.

   (a) [6 points] Show that $1/g_n$ is bounded for large n.

   (b) [7 points] Show that f_n/g_n converges uniformly on $[a,b]$ to f/g.

   (c) [7 points] Show that part (b) is false if $[a,b]$ is replaced by (a,b).

5. [15 points]

   (a) [7 points] State the definition of a compact subset of $\mathbb{R}^n$.

   (b) [8 points] Let $f:D\to\mathbb{R}^m$ be a continuous function on $D\subset\mathbb{R}^n$. If $K\subset D$ is a compact subset of $\mathbb{R}^n$, prove that f is uniformly continuous on K.

6. [15 points] Let $\{a_n\}_{n=0}^\infty$ be a decreasing sequence which converges to 0. Show that $\sum_{n=0}^\infty a_n$ is a convergent series if and only if the series $\sum_{n=0}^\infty a_n\cos nx$ is absolutely convergent for every real number x.