Linear Algebra

There are five problems with 100 points in this test.
Show your work for partial credits.

1. Let n be a positive integer and $P_n(\mathbb{R})$ consist of all polynomials with real coefficients and degree less than or equal to n. Define the transformation $\mathbb{T} : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by

$$\mathbb{T}(f(x)) = f'(x) + \int_0^x f(t)dt$$

where $f(x) \in P_2(\mathbb{R})$.

(a) Show that $\mathbb{T}$ is a linear transformation. (5 pt)

(b) Show that

$$\alpha = \{x, x^2 + x, 3 - 2x\} \text{ and } \beta = \{x, x^2 + x, 3 - 2x, x^3\}$$

are bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$, respectively. (5 pt)

(c) Let the vector space $P_2(\mathbb{R})$ be endowed with the following inner product:

$$< f, g >:= \int_{-1}^1 f(t)g(t)dt, \quad f, g \in P_2(\mathbb{R}).$$

Use the Gram-Schmidt process to replace the basis α by an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $P_2(\mathbb{R})$. (10 pt)

(d) Find the matrix of $\mathbb{T}$ relative to the bases α and β. (10 pt)

(e) Is $\mathbb{T}$ is one-to-one ? (5 pt)

2. Find a symmetric matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -5$, and corresponding eigenvectors $(1, 2, 3)^T$, $(-2, 1, 0)^T$, $(3, 6, -5)^T$. (10 pt)

3. Let the curve Γ be defined by

$$\Gamma := \left\{(x, y) \in \mathbb{R}^2 : \ 5x^2 + 4xy + 2y^2 - 10x - 4y - 1 = 0\right\}.$$

Find the maximum value of the function

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

where $(x_1, y_1), (x_2, y_2) \in \Gamma$, and determine the values of $(x_1, y_1), (x_2, y_2)$ for which this occur. (10 pt)

4. Determine which of the matrices

$$A_1 = \begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & -1 \\ -4 & 4 & -2 \\ -2 & 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}$$

are similar. (15 pt)

5. Prove or disprove the following statements. (30 pt)

(a) If A is an $m \times n$ matrix such that $(\text{row}(A))^{\perp} = \mathbb{R}^n$, then A must be the zero matrix.

(b) $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ for any $m \times n$ matrices A and B.

(c) Let A be an $m \times n$ matrix with linearly independent columns. Then AA^T is an invertible matrix.

(d) If A is a symmetry matrix satisfying $A^2 = 0$, then $A = 0$.

(e) Let A and B be two $n \times n$ matrices. If they are row equivalent, then A is similar to B.