There are four problems with 100 points.

Show your work for partial credits.

1. (35 pts) Let $A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$.

   a. Find the reduced row echelon form which is row equivalent to A.

   b. According to "a", find bases for the row space and column space of A, respectively.

   c. Find a basis for the null space of A.

   d. Let the vector space $\mathbb{R}^{1\times 5}$ be endowed with the standard inner product.

   Find an orthonormal basis for the row space of A.

2. (10 pts) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 1 & -2 \end{bmatrix}$. Find all the right inverses of A.

3. (20 pts) Let $\alpha \in \mathbb{R}$, $\vec{u}, \vec{v} \in \mathbb{R}^{n\times 1}$ and $A \in \mathbb{R}^{n\times n}$ be invertible. Define $B = \begin{bmatrix} A & \vec{u} \\ \vec{v}^T & \alpha \end{bmatrix}$.

   a. Show that if $\alpha \neq \vec{v}^T A^{-1} \vec{u}$, then B is invertible.

   b. If $\alpha = \vec{v}^T A^{-1} \vec{u}$, then $B\vec{x} = \vec{0}$ has nontrivial solutions.

4. (15 pts) Let $\vec{p} \neq \vec{0}$ be a fixed column vector in $\mathbb{R}^{n\times 1}$ and set $A = \vec{p}\vec{p}^T$.

   a. Prove that for each given $\vec{v} \in \mathbb{R}^{n\times 1}$, there exists a real constant α such that $A\vec{v} = \alpha\vec{p}$.

   b. Show that A is similar to $\begin{bmatrix} \theta & 0 & \cdots & 0 \\ 0 & 0 & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, where $\theta = \vec{p}^T\vec{p}$.

5. (20 pts) Prove or disprove the following statements.

   a. Let A and B be 2×2 real matrices. Then $AB = O$ implies $A = O$ or $B = O$.

   b. Let A and B be invertible $n \times n$ matrices. Then, if $A + B$ is invertible, we have $A^{-1} + B^{-1}$ is invertible.

   c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $(a-d)^2 + 4bc = 0$, then A is diagonalizable.

   d. Let $A \in \mathbb{R}^{n\times n}$ and $N(A)$ be the null space of A. Then $\dim N(A^2) \leqslant 2\dim N(A)$.