* There are 7 problems with 100 points in this test.
** Show your work for partial credits.

1. Let
$$V = M_{2\times2}(\mathbb{R}) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}$$
and
$$W_1 \equiv \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} \in V : a,b,c \in \mathbb{R} \right\},\ W_2 \equiv \left\{ \begin{pmatrix} 0 & a \\ a & b \end{pmatrix} \in V : a,b \in \mathbb{R} \right\}.$$
   (a) (5 pts) Prove that $W_1 + W_2$ is a subspace of V.
   (b) (5 pts) Find a basis for $W_1 + W_2$.

2. Let $P_n(\mathbb{R})$ be the space of all one-variable polynomials with real coefficients whose degree is at most n, that is,
$$P_n(\mathbb{R}) = \{f(x) = a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R},\ \text{for all } 0 \le i \le n\}.$$
Assume that the map $\Phi : P_n(\mathbb{R}) \longmapsto P_n(\mathbb{R})$ is given by $\Phi(f)(x) = f(x) - f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$. Show that:
   (a) (5 pts) Φ is linear.
   (b) (10 pts) Is Φ invertible? Justify your answer.

3. Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$.
   (a) (5 pts) Find the characteristic polynomial of A.
   (b) (5 pts) Find the minimal polynomial of A.
   (c) (5 pts) Let $f(t) = t^6 - 8t^5 + 19t^4 - 7t^3 - 28t^2 + 37t - 14$ and $f(A) = (b_{ij})_{3\times3}$. Find b_{21}.
   (d) (5 pts) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

4. (10 pts) Let A and B be $n \times n$ matrices over the complex number $\mathbb{C}$. Prove that there are at most n distinct scalars c in $\mathbb{C}$ such that $cA + B$ is not invertible.

5. (10 pts) Let A, B be 2×2 real matrices satisfying $AB = -BA$. Show that there exists a real number r such that $(AB)^2 = rI$. Here, I denotes the 2×2 identity matrix.

6. (15 pts) Let X, Y be finite dimensional linear spaces and $\{L(X,Y), +, \cdot\}$ denote the space of all linear transformations from X to Y, where the operations $+$ and $\cdot$ are given by
$$(f+g)(x) = f(x) + g(x)\ \&\ (c\cdot f)(x) = cf(x),\ \text{ for all } f,g \in L(X,Y)\ \&\ \text{for all scalars } c.$$
Assume that $V \subset X$ and $W \subset Y$ be linear subspaces of X and Y respectively. Express the dimension of the space
$$\{f \in L(X,Y) \mid f(V) \subset W\}$$
by the dimensions of X, Y, V, W.

7. Prove or disprove the following statements:

   (a) (5 pts) Let A, B be two $n \times n$ matrices such that $A + B$ is invertible. Then $(A+B)^{-1} = A^{-1} + B^{-1}$.

   (b) (5 pts) Let $M_{2\times 2}(\mathbb{R}) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ and $\langle A, B \rangle = \operatorname{tr}(A+B)$. Then $\langle \cdot, \cdot \rangle$ is an inner product on $M_{2\times 2}(\mathbb{R})$.

   (c) (5 pts) If $B \in M_{n\times n}(\mathbb{C})$ is skew-symmetric, that is, $B^T = -B$, and n is odd, then $\det B = 0$.

   (d) (5 pts) Let A be an $n \times n$ matrix with characteristic polynomial

   $$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

   If $a_0 \neq 0$, then A is invertible.