Advanced Calculus

1. (10 points) Let $A := [a, b) \subset \mathbb{R}$, where $b > a > 0$. Is A a closed set in $\mathbb{R}$? Why or why not? Is A a closed set in $[0, b)$? Why or why not? Justify your answer carefully.

2. Given the equations

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0; \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0. \end{cases}$$

   (a) (5 points) Is it possible to determine the function $u(x, y)$, $v(x, y)$ near $x = 2$, $y = -1$ such that $u(2, -1) = 2$, $v(2, -1) = 1$?

   (b) (5 points) Is it possible to find $\frac{\partial u}{\partial x}$? What is it?

3. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(a, b) \subset \mathbb{R}$ such that $f'(x) < 0$ for all $x \in (a, b)$. Prove that f is decreasing on $[a, b]$.

4. (20 points) Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and $F(x) = \int_a^x f(t)dt$. Show $F'(x) = f(x)$. Compute the derivative $\frac{d}{dx} \int_0^{x^{2010}} e^{\sqrt{t}} dt$.

5. (15 points) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable with

$$Df = c,$$

   where c is a constant. Show f has a linear term and a constant term such that the linear term has c as its coefficient.

6. (15 points) Show $\int_0^\infty e^{-xt} dt = \frac{1}{x}$ for $x > 0$. Basing on this fact, compute

$$\lim_{N \to \infty} \int_0^N \frac{\sin x}{x} dx.$$

7. (20 points) Let $f : [a, b] \to [0, \infty)$ be a nonnegative Riemann integrable function with $\int_a^b f(x)dx > 0$. Show that there is an $\epsilon > 0$ and two real numbers c, d such that $c < d$ and that $f(x) \geq \epsilon$ for all $x \in [c, d]$.