Linear Algebra

For the following problems, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively.

Please show all your work.

(1) **(15 Points)** Let

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 \\ 0 & -4 & 0 & 2 \\ 0 & 0 & -4 & 3 \end{pmatrix}$$

be a matrix with entries in $\mathbb{R}$. Find rank A and nullity A.

(2) **(18 Points)** Let V be the vector space spanned by $\cos x$ and $\cos 2x$ over $\mathbb{R}$. Find $\dim_{\mathbb{R}} V$.

(3) **(15 Points)** Let V be a complex vector space and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V over $\mathbb{C}$.

(a) Find a basis for V as a real vector space.

(b) Find $\dim_{\mathbb{R}} V$.

(4) **(18 Points)** Let V be a real inner product space.

(a) Let W be a subspace of V. In V we define

$$W^{\perp} = \{v \in V : v \perp w \text{ for all } w \in E\}.$$

Show that $W^{\perp}$ is a subspace of V.

(b) Let $W = \operatorname{Span}((1,1,-1,-1),\ (0,1,0,1))$. In the standard inner product space $\mathbb{R}^4$, find an orthonormal basis for $W^{\perp}$.

(5) **(18 Points)** Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

be a matrix with entries in $\mathbb{Q}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A over $\mathbb{Q}$.

(c) Is A diagonalizable over $\mathbb{Q}$? Explain!

(6) **(16 Points)** Let A be a 4×4 matrix with complex entries such that $A^3 = I$.

(a) Find rank A.

(b) Does A always have a Jordan form over $\mathbb{C}$? Explain!

(c) Give all *possible* Jordan forms of A over $\mathbb{C}$.