1. [15 points] Evaluate the integral

$$\int_0^\infty e^{-8x^2}\,dx.$$

2. [20 points] Define

$$f(x,y) = \begin{cases} \frac{x^3-xy^2}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

   (a) [7 points] Show that f is continuous at $(0,0)$.

   (b) [6 points] Show that f has directional derivatives in all directions at $(0,0)$.

   (c) [7 points] Show that f is not differentiable at $(0,0)$.

3. [15 points]

   (a) [7 points] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function and and $f'(a) < \alpha < f'(b)$. Show that there exists $c \in [a,b]$ such that $f'(c) = \alpha$.

   (b) [8 points] Let $g : [a,b] \to \mathbb{R}$ be a differentiable function. Use part (a) to show that if $g'(c-)$ and $g'(c+)$ exist for some $c \in [a,b]$, then $g'(c-) = g'(c+)$.

4. [20 points] Let f, g be two continuous functions defined on $[a,b]$ and $|g(x)| > 0$ for $x \in [a,b]$. Suppose that f_n and g_n converge uniformly on $[a,b]$ to f and g, respectively.

   (a) [6 points] Show that $1/g_n$ is bounded for large n.

   (b) [7 points] Show that f_n/g_n converges uniformly on $[a,b]$ to f/g.

   (c) [7 points] Show that part (b) is false if $[a,b]$ is replaced by (a,b).

5. [15 points]

   (a) [7 points] State the definition of a compact subset of $\mathbb{R}^n$.

   (b) [8 points] Let $f : D \to \mathbb{R}^m$ be a continuous function on $D \subset \mathbb{R}^n$. If $K \subset D$ is a compact subset of $\mathbb{R}^n$, prove that f is uniformly continuous on K.

6. [15 points] Let $\{a_n\}_{n=0}^\infty$ be a decreasing sequence which converges to 0. Show that $\sum_{n=0}^\infty a_n$ is a convergent series if and only if the series $\sum_{n=0}^\infty a_n \cos nx$ is absolutely convergent for every real number x.