Algebra and Linear Algebra

Instruction. *There are six problems and* 100 *points in total. The symbols* S_n, $\mathbb{Z}_n$, $\mathbb{Q}$, *and* $\mathbb{F}_q$ *stand for, respectively, the symmetric group of degree* n, *the cyclic group of order* n, *the field of rational numbers, and the finite field with* q *elements.*

1. [24%] In this problem, we consider groups of order 98.

(a) Show that every group of order 98 has a normal subgroup of order 49.

(b) Write down all abelian groups of order 98, up to isomorphism.

(c) In S_{14}, let $\alpha = (1,2,3,4,5,6,7)$ and $\beta = (8,9,10,11,12,13,14)$ be 7-cycles and $\gamma = (2,7)(3,6)(4,5)(9,14)(10,13)(11,12)$. Show that the subgroup $\langle \alpha, \beta, \gamma \rangle$ is non-abelian and has order 98.

(d) Find a non-abelian group of order 98 whose Sylow 7-subgroup is isomorphic to $\mathbb{Z}_{49}$.

2. [14%] Let $f(x) = x^4 + 1$. Find the splitting field and the Galois group of f over $\mathbb{Q}$.

3. [15%] Let $f(x) = x^5 + 3x^2 - 28x + 98$. Is f irreducible over $\mathbb{F}_3$? Over $\mathbb{F}_{243}$? Over $\mathbb{Q}$?

4. [15%] Let R be a finite ring with order $|R| > 1$. Suppose R has no zero divisors.

(a) If $a \in R$ is nonzero, prove that $aR = R$, where $aR = \{ar \mid r \in R\}$.

(b) Prove that R contains a (two-sided) identity, that is, there is an $e \in R$ such that $re = er = r$ for all $r \in R$.

(c) Prove that R is a division ring.

5. [16%] Let

$$A = \begin{bmatrix} -4 & 0 & -6 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}.$$

Find the Jordan canonical form of A and compute A^k for positive integers k.

6. [16%] Let A and B be n by n real symmetric matrices. Show that the eigenvalues of $AB - BA$ are of the form $r\sqrt{-1}$, where r is a real number.