Answer all questions. There are 7 problems with a total of 100 points.

(15%) 1. Let

$$f(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \text{if} \quad x \neq 0, \qquad \text{and} \quad f(0) = 0.$$

(i) Show that f is continuous at $x = 0$.
(ii) Show that f is differentiable at $x = 0$. Find $f'(0)$.

(15%) 2. Let

$$f(x,y) = \frac{x^2 y}{x^2 + y^2} \quad \text{if} \quad (x,y) \neq (0,0), \qquad \text{and} \quad f(0,0) = 0.$$

(i) Show that f is continuous at $(0,0)$.
(ii) Let θ be fixed, and $u = (\cos\theta, \sin\theta)$. Compute the directional derivative

$$\partial_u f(0) = \lim_{t \to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t}.$$

(iii) Show that f is not differentiable at $(0,0)$.

(15%) 3. Find the absolute maximum and absolute minimum of $f(x,y) = x^3 - x + y^2 - 2y$ on the closed triangular region with vertices at (0,0), (1,0), and (0,2).

(15%) 4. (i) Show that the equations

$$u^3 + xv - y = 0 \qquad \text{and} \qquad v^3 + yu - x = 0$$

can be solved for u and v as C^1 functions of x and y near the point $(x,y,u,v) = (0,1,1,-1)$.
(ii) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ at the point $(x,y,u,v) = (0,1,1,-1)$.

(15%) 5. Use the change of variables $x = u - uv$, $y = vu$ to evaluate the integral

$$\int\int_R \frac{1}{x+y}\, dA$$

where R is the region in the first quadrant between the lines $x + y = 1$ and $x + y = 4$.

(10%) 6. (i) Show that if the sequence $\{a_n\}$ is bounded, the radius of convergence of $\sum_{n=0}^{n} a_n x^n$ is least 1.
(ii) Suppose that the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is R. What is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^{2n}$? Why?

(15%) 7. (i) What is the definition of a disconnected set in $\mathbb{R}^2$?
(ii) Show that $S \subset \mathbb{R}^2$ is disconnected if and only if there is a continuous function $f : S \to \mathbb{R}$ such that $f(S)$ consists of the two points 0 and 1.