Notation: Let $M_{m\times n}(K)$ denote the set of matrices of size $m \times n$ over the field K and I_n the identity matrix of size $n \times n$. Let S_n be the symmetric group of degree n, and A_n the alternating group of degree n. Let $\mathbb{C}$ denote the field of complex numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ the finite commutative ring with n elements.

There are six problems and totally 100 points. You need to explain the details of your answers.

1. Consider $V = M_{2\times 2}(\mathbb{R}) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}\}$ as a vector space over $\mathbb{R}$. Let $\varphi : V \to V$ be a map from V to V such that $\varphi(A) = A^T$ for any $A \in V$, i.e., $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.
   (a) Find a basis of V and determine the dimension of V. (5 points)
   (b) Show that φ is a linear transformation from V to V. (5 points)
   (c) Determine the characteristic polynomial of φ. (5 points)
   (d) Find all eigenvalues of φ. (5 points)

2. Assume that W is a 4-dimensional vector space over $\mathbb{C}$ and $\{w_1, w_2, w_3, w_4\}$ is a basis of W. Let $T : W \to W$ be a linear transformation such that $T(w_1) = w_1 - w_2$, $T(w_2) = 2w_1 - w_2$, $T(w_3) = 2w_1 - w_2 + w_3 + w_4$, and $T(w_4) = 2w_1 - w_2 - w_3 - w_4$.
   (a) Determine the dimension of the kernel of T. (5 points)
   (b) Find the Jordan form of T. (10 points)
   (c) Determine the minimal polynomial of T^2. (8 points)
   (d) Determine if $T^2 : W \to W$ is diagonalizable. (7 points)

3. Let G be a group, $H \subseteq G$ a subgroup of G and $a \in G$.
   (a) Show that $aHa^{-1} = \{aha^{-1} \mid h \in H\}$ is a subgroup of G. (5 points)
   (b) Show that aHa^{-1} is isomorphic to H. (5 points)

4. Let G be a group with 255 elements and $H \subseteq G$ a subgroup of G with 17 elements. Show that H is a normal subgroup of G. (10 points)

5. (a) Determine if $\mathbb{Z}_{16}$ is a principal ideal domain. (5 points)
   (b) Give an example of a finite field with 16 elements. (10 points)

6. Let S_9 be the symmetric group of degree 9 and $\sigma = (12)(3457)(89)$, $\tau = (24)(1369)(57) \in S_9$.
   (a) Determine if σ is an even permutation or odd permutation. (5 points)
   (b) Find $\theta \in S_9$ such that $\tau = \theta\sigma\theta^{-1}$. (5 points)
   (c) Let $K = \{\rho\sigma\rho^{-1} \mid \rho \in S_9\}$. Determine how many elements there are in K. (5 points)