This exam has 7 questions, for a total of 100 points.

1. (10 points) Evaluate

$$\int_0^\infty e^{-2x^2}\,dx.$$

2. (a) (8 points) Suppose

$$f(x) = \int_x^{x^2} t/\log(t^2+1)\,dt,$$

find $f'(2)$.

   (b) (10 points) Find the function $y = f(x), x \in \mathbb{R}$ such that

$$y^2 x \frac{dy}{dx} - x^2 + 1 = 0$$

with initial condition $f(1) = 0$.

3. Suppose $\sum_{n=1}^\infty |a_n|$ is finite. Determine whether the following series is convergent or not and show your reasons.

   (a) (10 points) $\sum_{n=1}^\infty |a_n|^{1/2}$.

   (b) (10 points) $\sum_{n=1}^\infty a_n$.

4. (12 points) Denote

$$h(\alpha) = \begin{cases} \frac{1}{\lambda(\lambda+1)}\{\alpha^{1-\lambda} + (1-\alpha)^{1-\lambda} - 1\} & \text{if } \lambda \neq 0 \\ \frac{1}{2}\{-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha)\} & \text{if } \lambda = 0, \end{cases}$$

where $-1 < \lambda$ is a constant and $0.1 < \alpha < 1$. Find the maximum of h.

5. (10 points) Given $\mathbf{v}_1 = (-1,-2,-2), \mathbf{v}_2 = (0,1,4), \mathbf{v}_3 = (-1,1,2)$, and $\mathbf{w} = (3,1,2)$ in $\mathbb{R}^3$. Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

6. Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ be $n \times n$ matrices. Determine whether each of the following statements is true or false. If the statement is true, prove it. Otherwise, give a counterexample.

   (a) (5 points) If $\mathbf{AC} = \mathbf{BC}$, then $\mathbf{A} = \mathbf{B}$.

   (b) (5 points) If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $\mathbf{AB}$ is invertible.

7. Suppose $\mathbf{A} = \mathbf{I} - \frac{1}{n}\mathbf{J}'\mathbf{J}$ where $\mathbf{I}$ is an $n \times n$ identity matrix and $\mathbf{J} = (1,1,\ldots,1)$ is a $1 \times n$ vector, $n > 1$.

   (a) (10 points) Show that $\mathbf{A}$ is idempotent.

   (b) (10 points) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{A}$.