1. (10%) Let X_i be the amount of payment per insurance case which follows an exponential distribution with $E(X_i) = 4$. That is,

$$f_{X_i}(x) = \begin{cases} \frac{1}{4}e^{-\frac{1}{4}x} & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Let N be the number of insurance case and follow a Poisson distribution with $E(N) = 10$. Assume that X_i and N are independent.

(i) (5%) Find $E(X_1 + X_2 + \cdots + X_N)$.

(ii)(5%) Find $Var(X_1 + X_2 + \cdots + X_N)$.

2. (10%) Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution $N(0, \sigma^2)$. Now $n = 4$, let $U = (X_1, X_2, X_3, X_4)$, and $Y = ||U|| = \sqrt{X_1^2 + X_2^2 + X_3^2 + X_4^2}$.

(i) (3%) Assume $W \sim \chi^2_{(4)}$, find a c such that $F_Y(y) = Pr(W \leq c)$.

(ii)(7%) Based on (i), find the pdf of Y. Hint: Let $V \sim Gamma(\alpha, \beta)$, then

$$f_V(v) = \begin{cases} \frac{v^{\alpha-1}e^{-v/\beta}}{\Gamma(\alpha)\beta^\alpha} & 0 < v < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

3. (10%) There is a quick way to generate a simulated data from $N(0,1)$. The method is described as follows. "First, we use computer to generate 12 observations (say $X_1, X_2, \ldots, X_{12}$) from $U(0,1)$. Next, sum of these random numbers and subtract 6 and denote it by R. Finally, this obtained value seems to come from $N(0,1)$."

(i) (5%) Explain what is the theory behind this method in details. Specifically, why does R have an approximate distribution of $N(0,1)$.

(ii)(5%) Is the exact distribution of R really normal? Give the reasoning for your answer.

4. (15%)

(i) (3%) Give the definition of the MLE.

(ii) (5%) Let $X \sim Bin(1,p)$, $p \in [\frac{1}{5}, \frac{4}{5}]$. Find the MLE of p.

(iii) (7%) Consider a random sample from $Gamma(1, \beta)$. It is required to find the MLE of β in the following manner. A sample of size n is taken, and it is known only that k, $0 \leq k \leq n$, of these observations are $\leq M$, where M is a fixed positive number. Let $p = P(X_i \leq M) = 1 - e^{-M/\beta}$. Find the MLE of β.

5. (20%) Let $X_1, X_2, \ldots, X_n$ be i.i.d with pdf

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}e^{-\frac{1}{\theta}x} & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

(i) (4%) Find a uniformly minimum variance unbiased estimator (UMVUE) of θ.

(ii) (6%) Find a uniformly most powerful critical region of size $\alpha = 0.05$ and $n = 2$ for testing $H_0 : \theta = 2$ against $H_a : \theta > 2$. Show the details. ($\chi^2_{(2)} = 5.991, \chi^2_{(3)} = 7.815, \chi^2_{(4)} = 9.488, \chi^2_{(5)} = 11.071, \chi^2_{(6)} = 12.592$)

(iii) (6%) When $n = 3$ and let $Z_1 = X_1 + X_2 + X_3$, $Z_2 = X_2 + X_3$, $Z_3 = X_3$, find the joint pdf of Z_1, Z_2, Z_3.

(iv) (4%) Find $E(\frac{Z_1}{3}|z_3)$.

6. (20%) For the simple linear regression model, $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $i = 1, \ldots, n$ and ϵ_i are i.i.d $N(0, \sigma^2)$.

(i) (6%) To derive the MLE (Maximum Likelihood Estimator) for β_0, β_1 and σ^2.

(ii) (6%) Are $\hat{\beta}_0$ and $\hat{\sigma}^2$ unbiased estimators of β_0 and σ^2, respectively? Show the details.

(iii) (4%) Find $Var(\hat{\beta}_1)$.

(iv) (4%) Show the sum of squares regression (SSReg), $SSReg = \hat{\beta}_1^2 S_{xx}$, where $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$.

7. (15%) 有一種拍賣方式, 參加者每人將自已的出價金額寫在紙上, 以記名的方式投入一黑箱, 在所有人皆放入自已的紙條後, 最後再開箱唱名, 由出價最高者得標, 但是卻以次高價付款, 身爲拍賣會顧問的你, 想要了解得標者的付款金額的隨機現象, 因此你做了以下的數學假設:

假設 X_i 爲每人出價金額 (單位: 千萬), Y 爲得標者付款金額, 並假設每人出價金額彼此獨立且來自於相同的均勻分配 $U(0,1)$。今共有 15 人參與投標。

(i) (3%) 請問得標者的付款金額可以用統計學上的哪個統計量做描述?

(ii) (7%) 請求出 Y 的機率密度函數, 並說明此爲哪個著名的機率分配。

(iii) (5%) 請求出 Y 的期望值。