(10%) 1. Consider the linear transformation defined by

$$u(\boldsymbol{x}) = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_m - \bar{x} \end{bmatrix},$$

for all $\boldsymbol{x} \in R^m$, where $\bar{x} = (1/m)\Sigma x_i$. Find the matrix A for which $u(\boldsymbol{x}) = A\boldsymbol{x}$ and then determine the dimension of the range and null spaces.

(15%) 2. Let A be the 3×3 matrix given by

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(a) Find the eigenvalues and associated normalized eigenvectors of A.

(b) What is the rank of A?

(c) Find the eigenspaces and associated eigenprojections of A.

(20%) 3. Let S be the vector space spanned by the vectors $\boldsymbol{x}_1 = (1, 2, 1, 2)'$, $\boldsymbol{x}_2 = (2, 3, 1, 2)'$, $\boldsymbol{x}_3 = (3, 4, -1, 0)'$, and $\boldsymbol{x}_4 = (3, 4, 0, 1)'$.

(a) Find a basis for S.

(b) Use the Gram-Schmidt procedure on the basis found in (a) to determine an orthonormal basis for S.

(c) Find the orthogonal projection of $\boldsymbol{x} = (1, 0, 0, 1)'$ onto S.

(d) Find the component of $\boldsymbol{x}$ orthorgonal to S.

(10%) 4. Consider the set of vectors

$$\{(2, 1, 4, 3)', (3, 0, 5, 2,)', (0, 3, 2, 5)', (4, 2, 8, 6)'\}$$

(a) Show that this set of vectors is linearly dependent.

(b) From this set of four vectors find a subset of two vectors that is a linearly independent set.

(10%) 5. Find the points on the curve $5x^2 + 6xy + 5y^2 = 8$ that are closest to the origin.

(15%) 6. Draw a graph of the function $y = \dfrac{x^2(x-3)}{(x+3)^2}$. Find where the curve is increasing, decreasing, concave up and concave down, and list all the critical points and inflection points.

(10%) 7. Find the set of all number x for which the power series $\sum_{n=1}^{\infty} \dfrac{n!}{n^n} x^n$ converges.

(10%) 8. (a) Find the Taylor series expansion of $f(x) = e^{x^2}$ at $x = 0$.

(b) Find the approximate value of $\int_0^1 e^{x^2} dx$ up to third decimal place.