(15%) 1. Consider the $3 \times 3$ matrix

$$A = \begin{bmatrix} 9 & -3 & -4 \\ 12 & -4 & -6 \\ 8 & -3 & -3 \end{bmatrix}.$$

(a) Find the eigenvalues of $A$.

(b) Find a normaized eigenvector corresponding to each eigenvalue.

(c) Find the eigenvalues of $A'$.

(d) Determine the eigenspaces for $A'$ and compare these to those of $A$.

(16%) 2. Consider the $m \times m$ matrix $A = \alpha I_m + \beta \mathbf{1}_m \mathbf{1}_m'$, where $\alpha$ and $\beta$ are scalars, $I_m$ is the identity matrix of order $m$ and $\mathbf{1}_m$ is the $m \times 1$ vector having each component equal to 1.

(a) Find the eigenvalues and eigenvectors of $A$.

(b) Determine the eigenspaces and associated eigenprojections of $A$.

(c) For which values of $\alpha$ and $\beta$ will $A$ be nonsingular?

(d) Using (a), show that when $A$ is nonsingular, then

$$A^{-1} = \alpha^{-1} I_m - \frac{\beta}{\alpha(\alpha + m\beta)} \mathbf{1}_m \mathbf{1}_m'$$

(e) Show that the determinant of $A$ is $\alpha^{m-1}(\alpha + m\beta)$.

(8%) 3. Consider the vector space $S = \{\boldsymbol{u} : \boldsymbol{u} = A\boldsymbol{x}, \boldsymbol{x} \in R^4\}$, where $A$ is the $4 \times 4$ matrix given by

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 0 & 4 & 3 \\ 1 & 3 & -2 & 0 \end{bmatrix}$$

(a) Determine the dimension of $S$ and find a basis.

(b) Determine the dimension of the null space $N(A)$ and find a basis for it .

(20%) 4. Let $S$ be the vector space spanned by the vectors $\boldsymbol{x}_1 = (1, 2, 1, 2)'$, $\boldsymbol{x}_2 = (2, 3, 1, 2)'$, $\boldsymbol{x}_3 = (3, 4, -1, 0)'$, and $\boldsymbol{x}_4 = (3, 4, 0, 1)'$.

(a) Find a basis for $S$.

(b) Use the Gram-Schmidt procedure on the basis found in (a) to determine an orthonormal basis for $S$.

(c) Find the orthogonal projection of $\boldsymbol{x} = (1,0,0,1)'$ onto $S$,

(d) Find the component of $\boldsymbol{x}$ orthogonal to $S$.

(8%) 5. Consider the region $R$ in the $xy$-plane bounded by the graph of the equation

$$(x^2+y^2)^2 = 9(x^2-y^2).$$

(a) Convert the equation to polar coordinates.

(b) Use a double integral to find the area of the region $R$.

(8%) 6. Evaluate the multiple integral

$$\int_0^2 \int_{3y/2}^{5-y} e^{x+y}\,dxdy.$$

(10%) 7. We state without proof that

$$\begin{aligned} B(a,b) &= \int_0^1 x^{a-1}(1-x)^{b-1}dx \\ &= 2\int_0^{\pi/2} \cos^{2a-1} t \sin^{2b-1} t dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \end{aligned}$$

where $a>0$, $b>0$, and $B(a,b)$ is known as the beta function. Using this formula,

(a) show that

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\Gamma(1/n)}{n\Gamma[(n+2)/2n]}.$$

(b) evaluate

$$\int_0^{\pi/2} \sqrt{\tan x}dx.$$

(15%) 8. (a) Obtain the Taylor series of $\sin(3x^2)$ about $x=0$. What is the radius of convergence of the series?

(b) Expand the function

$$\int_x^1 \ln(1+x^3)dx$$

about $x=1$.

(c) Show that

$$1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1 \quad \text{for all } x.$$

Hint: Use Taylor's formula with Lagrange remainder for each of the inequalities.