(12%) 1. (a) Show the real matrix $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ is positive definite.

(b) Let A be a $n \times n$ real matrix. Show that there exists a symmetric matrix, B, such that $x^T A x = x^T B x$, $\forall x \in R^n$.

(14%) 2. Let $\varphi : R^2 \times R^2 \to R$ be defined by $\varphi(X, Y) = x_1 y_1 + x_2 y_2$, $X = (x_1, x_2)^T \in R^2$, $Y = (y_1, y_2)^T \in R^2$. Let $\alpha = \{\alpha_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\}$ be a basis of R^2. Further, there exists a Gram matrix of φ with respect to α, A, such that $\varphi(\sum_{i=1}^{2} x_i \alpha_i, \sum_{i=1}^{2} y_i \alpha_i) = (x_1, x_2) A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

(a) Find the Gram matrix of φ with respect to the basis, α.

(b) Use the result obtained from (a), to find $\varphi(\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix})$.

(13%) 3. Let $A = \begin{pmatrix} -2 & -1 & 3 \\ 6 & 5 & -6 \\ 2 & 2 & -1 \end{pmatrix}$.

(a) Find the eigenvalues of A.

(b) Find a non-singular matrix, P, such that $P^{-1} A P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

(14%) 4. Let A be a non-singular matrix.

(a) Show that if λ is an eigenvalue of A, then $\lambda \neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(b) Show that if $\lambda > 0$ is an eigenvalue of A^2, then either $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue of A.

(10%) 5. Let $f: R \to R$ is a differential function and satisfy $f(\sin(x)) = \tan(x)$.

Find $f'(\frac{\sqrt{3}}{2})$.

(15%) 6. Apply Taylor expression to compute the following limitation.

(a) $\lim_{x\to 0^+} \dfrac{e^x - (1 + x + \frac{x^2}{2})}{x^3}$

(b) $\lim_{x\to 0^+} \dfrac{\sqrt[3]{1+x} - (1 + \frac{x}{3} - \frac{x^2}{9})}{x^3}$

(c) $\lim_{x\to 0^+} \dfrac{\sin x - (x - \frac{x^3}{3!} + \frac{x^5}{5!})}{x^6}$

(10%) 7. Let $f(x) = \exp(-cx^2)$, $x \in R$ and c is a positive constant.

Find $\int_{-\infty}^{\infty} f(x)dx$.

(12%) 8. Evaluate the following limit.

(a) $\lim_{x\to 0} \dfrac{1}{x^2} \int_0^x t\sqrt{3 + t^5}\, dt$.

(b) $\lim_{x\to +\infty} e^{-x} \int_0^x e^{\sqrt{t}}\, dt$.