There are 6 problems with 100 points in this test.
Show your work for partial credits.

1. Let $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 2\\ 0 & 1 & 1\end{bmatrix}$ and $b=\begin{bmatrix}3\\0\\2\end{bmatrix}$.

   (1) (5 pts.) Solve the linear system $Ax=b$ by using LU factorization.

   (2) (10 pts.) Solve the linear system $Ax=b$ by using QR factorization

2. Let $A=\begin{bmatrix}6 & -5 & -7\\ 1 & 0 & -1\\ 3 & -3 & -4\end{bmatrix}$

   (1) (10 pts.) Diagonalize the matrix A.

   (2) (5 pts.) Find the particular solution of the system of differential equations

   $$\begin{bmatrix}y_1'\\ y_2'\\ y_3'\end{bmatrix}=A\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} \text{ with } y_1(0)=0,\ y_2(0)=2,\ y_3(0)=1.$$

3. Let $\begin{bmatrix}1 & 0 & 2 & 0 & 1\\ 0 & 1 & -1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$ be the reduced echelon form of matrix A. And, the first, third and fifth columns of A are $\begin{bmatrix}1\\-2\\0\\3\end{bmatrix}$, $\begin{bmatrix}0\\1\\3\\0\end{bmatrix}$, $\begin{bmatrix}5\\-8\\1\\2\end{bmatrix}$ respectively.

   (1) (4 pts.) Find the reduced echelon form of matrix $\begin{bmatrix}A & cA\end{bmatrix}$ for a nonzero scalar c.

   (2) (4 pts.) Find the matrix A.

   (3) (8 pts.) Let $T:\Re^5 \to \Re^4$ be a linear transformation defined by $T(x)=Ax$. Find the bases for the kernel and the range of T respectively.

4. Let $V = C([0,1])$ be the set of all continuous functions restricted in [0, 1] with the inner product defined by $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$, and let W be the subspace of V consisting of all polynomial functions of degree less than or equal to 2 with domain restricted on [0, 1].
(1) (10 pts.) Find an orthonorormal basis for W.
(2) (5 pts.) Find the orthogonal projection of the function $f(t) = \sqrt{t}$ on W.

5. Let $B = \{1, x, x^2, x^3\}$ be a basis for P_3 which is the vector space consisting of all polynomials of degree less than or equal to 3. And, $T : P_3 \to P_4$ is defined by $T(x^k) = \int_0^x t^k dt$.
   (1) (5 pts.) Show that T is linear.
   (2) (10 pts.) Find the matrix A for T with respect to the basis B for P_3 and the basis $B' = \{1, x-1, x^2+x+1, x^3+1, x^4-1\}$ for P_4.

6. (24 pts.) State true or false for each statement, and prove or disprove it.
   (1) If u and v are vectors in an inner product space and $\langle u, v\rangle^2 = \langle u, u\rangle\langle v, v\rangle$, then $\{u, v\}$ is a linearly dependent set.
   (2) There is a 2×2 matrix B such that $T : M_{22} \to M_{22}$ defined by $T(A) = AB - BA$ is an isomorphism.
   (3) If $x'Ax$ is a quadratic form with no cross product terms, then A is a diagonal matrix.
   (4) If A is a skew-symmetric matrix with $A^T = -A$, then $\det A = (-1)^n \det A$.