

(20%) 1. Let f be a real-valued function defined on $[a, b]$ and $x_0 \in [a, b]$. Show that the following two properties are equivalent.

- (a) for any $\varepsilon > 0$, there exists a constant $\delta > 0$ so that for $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$, we have $|f(x) - f(x_0)| < \varepsilon$.
- (b) for any sequence $\{c_n\}_{n=1}^{\infty} \subset [a, b]$ satisfying $\lim_{n \rightarrow \infty} c_n = x_0$, we have $\lim_{n \rightarrow \infty} f(c_n) = f(x_0)$

(20%) 2. Let $\{a_n\}_{n=1}^{\infty}$ be a real-valued sequence satisfying

$$\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 0.$$

Set $f(x) = \sum_{n=1}^{\infty} a_n x^n$, for $x \in (-\infty, \infty)$. Show that

- (a) f is continuous in $(-\infty, \infty)$, and
- (b) the derivative f' is also continuous in $(-\infty, \infty)$ and we have

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \text{ for } x \in (-\infty, \infty).$$

(20%) 3. Let $\{a_n\}_{n=1}^{\infty}$ be a real-valued sequence and $\sum_{n=1}^{\infty} |a_n|$ exist. Set $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$, for $x \in [-\pi, \pi]$. Show that

- (a) the function f is continuous on $[-\pi, \pi]$, and
- (b) the integral

$$\int_{-\pi}^{\pi} f^2(x) dx = \pi \sum_{n=1}^{\infty} a_n^2.$$

(20%) 4. Show the following

(a) for $\varepsilon \in (0, 1]$, we have $\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{100}} \ln \varepsilon = 0$,

(b) $\lim_{x \rightarrow \infty} \frac{x^{100}}{e^x} = 0$,

and find the value of

(c) $\int_0^1 x^{-\frac{99}{100}} \ln x dx = ?$,

(d) $\int_0^{\infty} e^{-x} \cos(5x) dx = ?$.

(20%) 5. Let f be a bounded variation function defined on $[a, b]$. Show that

(a) there exist two increasing functions g and h defined on $[a, b]$ so that

$$f = g - h,$$

and

(b) f is continuous on $[a, b]$ except possibly countable many points.

There are six problems and 100 points in total.

1. (12 pts.) Let W be the subspace of \mathbb{R}^3 spanned by $\mathbf{a}_1 = [1, 1, -1]$, $\mathbf{a}_2 = [0, 1, -2]$, $\mathbf{a}_3 = [2, 3, -4]$, and $\mathbf{a}_4 = [0, 3, -6]$.

- (a) Determine whether the vector $\mathbf{b} = [1, -1, 3]$ lies in W .
(b) Find a basis for W .

2. (18 pts.) Let $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & -3 & 1 \\ 4 & 1 & -3 & 0 \end{bmatrix}$.

- (a) Determine the rank of A .
(b) Find a basis for the column space and a basis for the row space of A .
(c) Find a basis for the nullspace of A .

3. (18 pts.) Let A and B be $n \times n$ matrices.

- (a) Prove that $\text{rank}(AB) \leq \text{rank}(A)$.
(b) Prove that $\text{rank}(AB) \leq \text{rank}(B)$.
(c) Find a necessary and sufficient condition for $\text{rank}(AB) = \text{rank}(B)$.

4. (12 pts.) Let T be the linear operator on the vector space $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ given by

$$T(f(x)) = f(1) + f'(0)x + 2f''(0)x^2.$$

- (a) Give the matrix of T relative to the standard basis $\{1, x, x^2\}$ of P_2 .
(b) Is T diagonalizable?

5. (20 pts.) Suppose that a 4×4 matrix A has eigenvalues 1, -1 , 2, and -2 . Find the trace and determinant of $A^6 - 5A^4 + 4I$.

6. (20 pts.) Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU =$

$$D, \text{ where } A = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Show all your work.

- (10 pts.) A light is 4 miles from a straight shoreline. The light revolve at the rate of 2 rev/min. Find the speed of the spot of light along the shore when the light spot is 2 miles past the point on the shore closest to the source of light.
- (10 pts.) Sketch the graph of $f(x) = e^{-x} \sin x$ for $x \geq 0$, and determine as many as possible of the key features such as range, intercepts, relative extrema, inflection points, asymptotes, and concavity.
- (10 pts.) Minimize $f(x, y, z) = 2x^2 + 3y^2 + 4z^2$ subject to $x + y = 4$ and $x - 2y + 5z = 3$.

- (10 pts.) Find the equation for the tangent line to the curve

$$y = F(x) = \int_1^{\sqrt{x}} \frac{t^2 + t + 1}{\sqrt{3t^2 + 1}} dt \quad \text{at } x = 1.$$

- (10 pts.) Find the volume of the solid D bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $2x + z = 3$.

- (10 pts.) Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

(1) Use the definition of the derivative to prove that f is differentiable at $x = 0$.

(2) Prove or disprove that $f'(x)$ is continuous at $x = 0$.

- (10 pts.) Let $a_n = \left(1 + \frac{1}{n}\right)^n, n = 1, 2, 3, \dots$

(1) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

(2) Approximate $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ correct to four decimal places.

- (10 pts.) Let $f(x, y, z) = z(x - y)^5 + xy^2z^3$.

(1) Find the directional derivative of f at $(2, 1, -1)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 6$.

(2) In what direction is the directional derivative at $(2, 1, -1)$ largest?

- (20 pts.) Define $F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$. And, the inverse $L^{-1}\{F(s)\}$ is the function $f(t)$ such that $L\{f(t)\} = F(s)$.

(1) Show that $L\{e^{at} f(t)\} = F(s - a)$ and $L\{tf(t)\} = -F'(s)$.

(2) Find $L\{t^n\}$, $L\{\cos at\}$ (with $s - a > 0$), $L\{t \cos 2t\}$, and $L^{-1}\left\{\frac{5}{s}\right\}$.

For the following problems, P_n denotes the vector space of all polynomials with real coefficients of degree less than or equal to n , V^\perp denotes the orthogonal complement of vector space V , and A^T denotes the transpose of the matrix A .

- Let $V = \{[x_1, x_2] \mid x_1 \text{ is a real number and } x_2 \text{ is a positive number}\}$ with addition defined by $[x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1 + 1, x_2 y_2]$ and with scalar multiplication defined by $r[x_1, x_2] = [rx_1 + r - 1, (x_2)^r]$. (for example $[1, 3] \oplus [-3, 7] = [-1, 21]$, $2[1, 3] = [2 + 2 - 1, 3^2] = [3, 9]$)
 (a) Find the additive identity $\vec{0}$ and the additive inverse of $\vec{v} = [3, 2]$. (b) Show the scalar multiplication satisfies the distributive property $r(\vec{x} \oplus \vec{y}) = r\vec{x} \oplus r\vec{y}$. (13%)
- Find a basis for the subspace $V = \{p(x) \mid p(x) = x^4 p(\frac{-1}{x}) \text{ for } p(x) \in P_4\}$. (7%)
- Let $V = \{p(x) \mid p(x) \text{ is a polynomial without constant term}\}$ be the subspace of polynomial space P with inner product $\langle p(x), q(x) \rangle = \int_0^1 xp(x)q(x) dx$. Show that $V^\perp = \{0\}$. (5%)
- The matrix A is row-equivalent to the matrix B .

$$A = \begin{bmatrix} 1 & a_1 & 1 & 0 & b_1 \\ -1 & a_2 & 1 & 0 & b_2 \\ 3 & a_3 & -1 & 0 & b_3 \\ 1 & a_4 & 0 & 1 & b_4 \\ 2 & a_5 & 3 & -1 & b_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Find a basis for the nullspace of A . (5%)
 - Find vectors $\vec{a} = [a_1, a_2, \dots, a_5]^T$ and $\vec{b} = [b_1, b_2, \dots, b_5]^T$. (5%)
 - Find bases for the row space and the column space of A , respectively. (5%)
- Let \vec{v} be a column vector in R^n . Define the matrix $A = I - \alpha \vec{v} \vec{v}^T$. Find the value of α so that $A^{-1} = A$. Solve the linear system $A\vec{x} = \vec{b}$ where $\vec{v} = [1, 0, 2, 0, -1]^T$ and $\vec{b} = [0, 11, -1, 9, 1]^T$. (10%)
 - Let $\{\phi_0(x), \phi_1(x), \dots, \phi_n(x)\}$ be an orthogonal basis for P_n with a given inner product denoted by $\langle \cdot, \cdot \rangle$. If each $\phi_k(x)$ is a monic (the leading coefficient is 1) polynomial of degree k , show that $\phi_{k+1}(x) - x\phi_k(x) = -\alpha_k \phi_k(x) - \beta_k \phi_{k-1}(x)$, where $\alpha_k = \frac{\langle x\phi_k(x), \phi_k(x) \rangle}{\langle \phi_k(x), \phi_k(x) \rangle}$ and $\beta_k = \frac{\langle \phi_k(x), \phi_k(x) \rangle}{\langle \phi_{k-1}(x), \phi_{k-1}(x) \rangle}$ for $k = 1, 2, \dots, n-1$. (8%)
 - Let $\{1, x, x^2 - \frac{1}{3}, x^3 - \frac{2}{5}x, \phi_4(x)\}$ be an orthogonal basis for P_4 with respect to the inner product $\langle p(x), q(x) \rangle = \int_{-1}^1 p(x)q(x) dx$. Find $\phi_4(x)$. (7%)
 - Consider the vector space P_2 of polynomials of degree at most 2, and let $T: P_2 \rightarrow P_2$ be the linear transformation such that $T(x^2 - 1) = -x^2 + 1$, $T(x) = x^2 - x - 1$ and $T(1) = x^2 - 3x + 1$.
 (a) Find a matrix representation for T associated with the ordered basis $\{x^2 - 1, x, 1\}$. (5%)
 (b) Find $p(x)$ such that $T(p(x)) = x + 2$. (5%)
 (c) Find eigenvalues λ and the associated eigenfunctions $p(x)$ for T . (i.e. $T(p(x)) = \lambda p(x)$) (5%)
 (d) Find $T^8(x^2 - 2x + 1)$ (5%)
 - Determine whether the statement is true or false. If it is true, prove it, otherwise, give a counter example. (15%)
 (a) If S is a subspace of an inner product space V , then $(S^\perp)^\perp = S$.
 (b) Let $T: R^n \rightarrow R^n$ be a linear transformation. Let the set $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ be linearly independent in R^n , then the set $\{T(\vec{x}_1), T(\vec{x}_2), \dots, T(\vec{x}_k)\}$ is also linearly independent.
 (c) Let \vec{v} be a unit column vector in R^n . The characteristic polynomial $(\det(xI - A))$ of the matrix $A = \vec{v}\vec{v}^T$ is $x^{n-1}(x - 1)$.

(10%) 1. Let $f: (0, +\infty) \rightarrow \mathbb{R}$ be differentiable and $\lim_{x \rightarrow +\infty} f'(x) = 5$.

Find $\lim_{x \rightarrow +\infty} (f(x+5) - f(x))$.

(20%) 2. Evaluate the following limit:

(5%) (a) $\lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$.

(5%) (b) $\lim_{x \rightarrow 1^-} \frac{\frac{\pi}{2} - \sin^{-1}(x)}{x-1}$.

(5%) (c) $\lim_{x \rightarrow 0^+} x^{\frac{1}{2+\ln(x)}}$.

(5%) (d) $\lim_{x \rightarrow -\infty} x^2(1 - x \sin(\frac{1}{x}))$.

(20%) 3. Evaluate the following integral:

(5%) (a) $\int_2^{41+x} \frac{1}{1-x} dx$.

(5%) (b) $\int_{\pi/4}^{\pi/2} \frac{\cos^3(x)}{\sqrt{\sin(x)}} dx$.

(5%) (c) $\int_1^e \ln^2(x) dx$.

(5%) (d) $\int_0^2 \frac{1}{1+e^x} dx$.

(10%) 4. Let $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx$, $\alpha \in (0, +\infty)$. Prove

(5%) (a) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\forall \alpha > 0$.

(5%) (b) $\Gamma(n) = (n-1)!$, $\forall n \in \mathbb{N}$.

(20%) 5. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$.

(5%) (a) Find A^{-1} .

(15%) (b) Find the eigenvalues of A and a non-singular matrix, P , such

$$\text{that } P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

(10%) 6. Show that if $A^2 = A$, and if λ is an eigenvalue of A , then either $\lambda = 1$ or $\lambda = 0$.

(10%) 7. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Find a symmetric matrix, B , such that $x^T Ax = x^T Bx$,
 $\forall x \in R^2$.

1. (35%) Let X_1, \dots, X_n be iid with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 \leq x \leq 1, 0 < \theta < \infty, \\ 0 & \text{elsewhere} \end{cases}$$

- (i) (7%) Find the maximum likelihood estimator (MLE) of θ . Is the MLE of θ unbiased? Show the details.
- (ii) (4%) Find the Rao-Cramer's lower bound of MLE of θ .
- (iii) (3%) Let W be Beta(1,1), and consider $Y = -\theta \log W$, where $\theta > 0$. Find the cdf of Y .
- (iv) (6%) Suppose the random sample Y_1, \dots, Y_n from the distribution of Y . Find the MLE and the minimum variance unbiased estimator (MVUE) of $P(Y \leq 2)$.
- (v) (5%) Suppose the random sample Y_1, \dots, Y_n from the distribution of Y . Find a uniformly most powerful critical region of size $\alpha = 0.05$ and $n = 2$ for testing $H_0 : \theta = 2$ against $H_a : \theta > 2$. Show the details. ($\chi_{(2)}^2 = 5.991, \chi_{(3)}^2 = 7.815, \chi_{(4)}^2 = 9.488, \chi_{(5)}^2 = 11.071, \chi_{(6)}^2 = 12.592$)
- (vi) (4%) Let the random sample W_1, \dots, W_n from Beta(1,1) and let $W_{(1)}, \dots, W_{(n)}$ denote the order statistics. Find the distribution of the j^{th} order statistic $W_{(j)}$. Show the details.
- (vii) (6%) Let the random sample W_1, \dots, W_n from Beta(1,1) and let $W_{(1)}, \dots, W_{(n)}$ denote the order statistics. The range is defined as $R = W_{(n)} - W_{(1)}$ and the mid-range is defined as $V = (W_{(1)} + W_{(n)})/2$. Find the distribution of R .

2. (8%) Consider X and Y have a trinomial distribution with joint pmf

$$p(x, y) = \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y p_3^{n-x-y},$$

where x and y are nonnegative integers with $x + y \leq n$;

$p_1, p_2, p_3 \in (0, 1)$ and $p_1 + p_2 + p_3 = 1$; and let $p(x, y) = 0$ elsewhere.

- (i) (4%) Find the moment generating function of a trinomial distribution.
- (ii) (4%) Compute $E(Y|X = x)$.
3. (13%) Let $X \sim N(0, \theta)$ where $0 < \theta < \infty$.
- (i) (3%) Show that the family $N(0, \theta)$ where $0 < \theta < \infty$ is not complete by finding at least one nonzero function $u(x)$ such that $E(u(X)) = 0$, for all $\theta > 0$.
- (ii) (6%) If X_1, X_2, \dots, X_n be a random sample from $N(0, \theta)$, show that the MLE of θ is an efficient estimate of θ .
- (iii) (4%) Find the minimum variance unbiased estimator of θ^2 .
4. (20%) For the simple linear regression model, $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $i = 1, \dots, n$ and ϵ_i are i.i.d $N(0, \sigma^2)$.
- (i) (4%) To derive the least squared estimator(s) (LSE) for β_0 and β_1 .
- (ii) (5%) Find the sampling distribution of $\hat{\beta}_1$. Show the details.
- (iii) (5%) For testing $H_0 : \beta_1 = 0$. Find the distribution of $SSReg/\sigma^2$ under H_0 is true, where $SSReg$ is the sum of squares regression.
- (iv) (6%) Derive the likelihood ratio test for testing $H_0 : \beta_1 = 0$ against $H_a : \beta_1 \neq 0$.

5. (24%) Let X_1, \dots, X_n be a random sample from a distribution, where $Pr(X_i = 1) = \theta$, $Pr(X_i = 0) = 1 - \theta$, where $0 < \theta < 1$.

(i) (5%) Show that this is a uniformly most powerful test when we test $H_0 : \theta = \frac{1}{2}$ against $H_a : \theta < \frac{1}{2}$.

(ii) (5%) Based on (i). Use the Central Limit Theorem (CLT) to find n and c so that the significance level is approximately 0.10 and the power of the test is approximately 0.80 when $\theta = \frac{1}{3}$.

(iii) (6%) If the prior p.d.f of Θ is

$$f(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}, & 0 < \theta < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \sum_{i=1}^n X_i$, find the posterior p.d.f, $f(\theta|Y)$.

(iv) (3%) Take the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$; find the Bayes' solution $\delta(y)$ for a point estimate of θ .

(v) (5%) Find $E(Y)$ and $Var(Y)$.