There are six problems and 100 *points in total.*

1. (20 pts.) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \\ -2 & 0 & -1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find the characteristic and minimal polynomials of A.

(c) Find A^n for $n \geq 1$.

2. (20 pts.) Let A be an $m \times n$ matrix over $\mathbb{R}$. Let A^t, $R(A)$, and $N(A)$ denote the transpose, range, and null space of A, respectively. Show that

(a) $R(A) = N(A^t)^\perp$, the orthogonal complement of $N(A^t)$ in $\mathbb{R}^m$.

(b) $N(A^tA) = N(A)$.

(c) A^tA and A have the same rank.

3. (18 pts.) Suppose that V is a vector space and $T\colon V \to V$ is linear. Show that if V is finite-dimensional, then T is invertible if and only if T is one-to-one. What if V is infinite-dimensional?

4. (18 pts.) Let $V = \mathbb{C}^n$ and $S\colon V \to V$ be a linear map given by $S(\mathrm{e}_1) = \mathrm{e}_n$ and $S(\mathrm{e}_i) = \mathrm{e}_{i-1}$ for $i = 2, 3, \ldots, n$, where $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$ is the standard basis of V.

(a) Prove that there is a basis $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ of V and $\alpha_i \in \mathbb{C}$ such that $S(\mathrm{v}_i) = \alpha_i \mathrm{v}_i$ for $i = 1, 2, \ldots, n$.

(b) Give these α_i's explicitly.

(c) Can this happen with $V = \mathbb{R}^n$ and all $\alpha_i \in \mathbb{R}$?

5. (12 pts.) Let T be a linear operator on $P_2(\mathbb{R}) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ defined by $T(f(x)) = f(x) - 2f'(x)$.

(a) Find the matrix representation of T with respect to the standard basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$.

(b) Find a Jordan canonical form of T.

6. (12 pts.) Evaluate the following $n \times n$ determinant

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} & a_2^n \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-2} & a_3^n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-2} & a_n^n \end{vmatrix}.$$