Advanced Calculus

This exam contains 7 problems with total 100 points.
To earn partial credits, show your work.

1. (10 pts) Prove it if the statement is true. Give a counterexample if the statement is false.

   (a) Let $a, b \in \mathbb{R}$ and f, g be two real-valued functions. Assume that $f : [a, b] \longmapsto \mathbb{R}$ is continuous and nonnegative in the interval $[a, b]$ and that $g : (a, b) \longmapsto (a, b)$ is differentiable and increasing in the interval (a, b). Then the function $F(x) = \int_a^{g(x)} f(t)dt$ is increasing on the interval (a, b).

   (b) The function $f(x, y) = \sqrt{|xy|}$ is differentiable at the point $(0, 0)$.

2. (10 pts) Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and $a < b$ and let f be a real-valued function defined on the interval (a, b). Assume that f is differentiable on (a, b) and that f' is bounded on (a, b). Prove that f is uniformly continuous on (a, b).

3. (20 pts) Let $f : [a, b] \to \mathbb{R}$ be an increasing function defined on the closed interval $[a, b]$.

   (a) Let $P = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[a, b]$ and let $M_i(f) = \sup f([t_{i-1}, t_i])$, $m_i(f) = \inf f([t_{i-1}, t_i])$ and $\Delta t_i = t_i - t_{i-1}$, for $i = 1, \cdots, n$. Show that

   $$\sum_{i=1}^{n}(M_i(f) - m_i(f))\Delta t_i \leq (f(b) - f(a))\|P\|,$$

   where $\|P\| = \max_{1 \leq i \leq n} \Delta t_i$.

   (b) Show that f is integrable on $[a, b]$.

4. (20 pts)

   (a) Show that $e^x \geq 1 + x \geq e^{\frac{x}{1+x}}$ for all $x \geq 0$.

   (b) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Prove that:

   $$\prod_{n=1}^{\infty}(1 + a_n) \text{ converges if and only if } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

5. (15 pts)

   (a) Prove that the improper integral $\int_0^\infty e^{-x^2}dx$ converges to a finite real number.

   (b) Let $I = \int_0^\infty e^{-x^2}dx$. Show that $I^2 = \lim_{N\to\infty} \int_0^{\pi/2} \int_0^N e^{-r^2} r dr d\theta$.

   (c) Show that $\displaystyle\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

6. (10 pts) Prove that the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \cdot \ln n \cdot [\ln(\ln n)]^p}$ converges if and only if $p > 1$.

7. (15 pts) Let f be a real-valued function. Assume that f is differentiable at a point $\vec{c} \in \mathbb{R}^n$.

   (a) Let $\vec{u}$ be any unit vector. Show that the directional derivative of f at $\vec{c}$ in the direction $\vec{u}$ exists.

   (b) Let $f'(\vec{c};\vec{u})$ denote the directional derivative of f at $\vec{c}$ in the direction $\vec{u}$. Show that $f'(\vec{c};\vec{u}) = \nabla f(\vec{c}) \cdot \vec{u}$ for any unit vector $\vec{u} \in \mathbb{R}^n$.

   (c) Assume that $\|\nabla f(\vec{c})\| \neq 0$, where $\|\cdot\|$ denotes the length of a given vector. Prove that there exists one and only one unit vector $\vec{u} \in \mathbb{R}^n$ such that $f'(\vec{c},\vec{u}) = \|\nabla f(\vec{c})\|$ and that this is the unit vector for which $f'(\vec{c},\vec{u})$ attains its maximum.