(15%) 1. Find the following limit:

(5%) (a) $\lim_{x\to 0}(\frac{1}{\sin(x)}-\frac{1}{x})$

(5%) (b) $\lim_{x\to\infty} x^{1/x}$

(5%) (c) $\lim_{x\to\infty}\tan^{-1}(\frac{1+x^2}{1+x})$

(10%) 2. Let f and g be differentiable and satisfy $f(g(x))=x^2$, $x\in R$, and $f'(x)=1+(f(x))^2$. Find $g'(x)$.

(10%) 3. Apply Taylor's theorem to prove that if $x>0$, then

$$1+\frac{x}{3}-\frac{x^2}{9}<\sqrt[3]{1+x}<1+\frac{x}{3}.$$

(15%) 4. Evaluate the following integral:

(5%) (a) $\int_1^5 \frac{x}{1+x^2}dx$

(5%) (b) $\int_1^e \sin(\ln(x))dx$

(5%) (c) $\int_0^1(\int_x^1 \tan^{-1}(y)dy)dx$

(10%) 5. Evaluate the following series:

(5%) (a) $\sum_{k=1}^{\infty}(\frac{5}{2^k}-\frac{1}{k(k+1)})$

(5%) (b) $\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+2)}$

(10%) 6. Let A be an orthonormal matrix. Find the determinant of A.

(10%) 7. Let $T: R^2 \to R^2$, $T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$.

(5%) (a) Find the standard matrix for T.

(5%) (b) Find the matrix for T relative to the basis $B' = \{(1,0),(1,1)\}$.

(10%) 8. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP$ is diagonal.

(10%) 9. Prove that if A and B are similar $n \times n$ matrices (i.e. exist an invertible matrix P such that $B = P^{-1}AP$), then they have the same eigenvalues.